CHAPTER 1

MATHEMATICAL LOGIC

1.1 Fundamentals of Mathematical Logic

Logic is commonly known as the science of reasoning. Some of the reasons to study logic are the following: At the hardware level the design of 'logic' circuits to implement instructions is greatly simplified by the use of symbolic logic. At the software level knowledge of symbolic logic is helpful in the design of programs.

1.2 Propositions and Related Concepts

A proposition is any meaningful statement that is either true or false, **but not both**. We will use lowercase letters, such as p; q; r; to represent propositions. We will also use the notation

p: 1 + 1 = 3

to define p to be the proposition 1 + 1 = 3

The truth value of a proposition is true, denoted by T, if it is a true statement and false, denoted by F, if it is a false statement. Statements that are not propositions include questions and commands.

Example 1.2.1

Which of the following are propositions? Give the truth value of the propositions.

a. 2 + 3 = 7

b. Julius Caesar was president of the United States.

c. What time is it?

d. Be quiet !

e. The difference of two primes.

1.3 Connectives and Truth Values

Many propositions are the combination of other simple propositions with connecting words. The two most common connecting words are '**AND**' and '**OR**'. The propositions that form a propositional function are called the propositional variables. Let p and q be propositions.

The ***conjunction*** of p and q; denoted $p \wedge q$; is the proposition: p and q:

This proposition is defined to be true only when both p and q are true and it is false otherwise.

The ***disjunction*** of p and q; denoted $p \vee q$; is the proposition: p or q:

The 'or' is used in an inclusive way. This proposition is false only when both p and q are false, otherwise it is true.

Example 1.3.1: Let p : 5 < 9 , q : 9 < 7

Construct the propositions $p \wedge q$ and $p \vee q$

Example 1.3.2

Consider the following propositions

p : It is Friday

q : It is raining

Construct the propositions $p \wedge q$ and $p \vee q$:

A truth table displays the relationships between the truth values of propositions.

Next, we display the truth tables of $p \wedge q$ and $p \wedge q$:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Let p and q be two propositions. The ***exclusive*** **or** of p and q; denoted p $\oplus$ q; is the proposition that is true when exactly one of p and q is true and is false otherwise. The truth table of the exclusive 'or' is displayed below

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example 1.3.3

a. Construct a truth table for $(p \oplus q) \oplus r$

b. Construct a truth table for $p \oplus p$

Next operation on a proposition p that we discuss is the ***negation*** of p: The negation of p; denoted $\sim p$; is the proposition not p: The truth table of $\sim p$ is displayed below

p	$\sim p$
T	F
F	T

Example 1.3.4

p : " 6 is a prime number"

$\sim p$: " 6 is not a prime number"

1.4 Complex Propositions:

Example 1.4.1

Consider the following propositions:

p: Today is Thursday.

q: 2 + a = 3:

r: There is no pollution in New Jersey.

Construct the truth table of [~ (p $\wedge$ q)] $\vee$ r.

Definition 1.1 A compound proposition is called a ***tautology*** if it is always true, regardless of the truth values of the basic propositions which comprise it.

Definition 1.2 A proposition, which is always false, no matter what truth values are assigned to its component proposition, is called **contradiction**.

Example 1.4.2

a. Construct the truth table of the proposition (p $\wedge$ q) $\vee$ $(\sim p \vee \sim q)$. Determine if this proposition is a tautology.

b. Show that p $\vee$ $\sim p$ is a tautology.

Thus, the given proposition is a tautology.

Again, this proposition is a tautology.

Two propositions are equivalent if they have exactly the same truth values under all circumstances. We write p $\equiv$q:

Example 1.4.3

a. Show that ~ (p $\vee$ q) $\equiv$ ~ p $\wedge$ ~ q
b. Show that ~ (p $\wedge$ q) $\equiv$ ~ p $\vee$ ~ q
c. Show that ~ (~ p) $\equiv$ p

a. and b. are known as DeMorgan's laws.

Example 1.4.4

a. Show that p $\wedge$ q $\equiv$ q $\wedge$ p and p $\vee$ q $\equiv$ q $\vee$ p
b. Show that (p $\vee$ q) $\vee$ r $\equiv$ p $\vee$ (q $\vee$ r) and (p $\wedge$ q) $\wedge$ r $\equiv$ p $\wedge$ (q $\wedge$ r)
c. Show that (p $\wedge$ q) $\vee$ r $\equiv$ (p $\vee$ r) $\wedge$ (q $\vee$ r) and (p $\vee$ q) $\wedge$ r $\equiv$ (p $\wedge$ r) $\vee$ (q $\wedge$ r)

Example1.4.5:

Let $p : x < 5$ $q: 2x < 10.7$ $r: SQRT\ (5x) > 5.1$

Statement $[(x < 5)\ \ AND\ (2x < 10.7)]\ \ OR\ \ [SQRT\ (5x) > 5.1]$

Example 1.4.6:

If $\left\{\{((X < Y)\ \ AND\ \ (Y = 2))\ OR\ \ (z = 10)\}\ \ AND\ \ \{NOT\ ((X < Y)\ AND\ (Z = 10))\}\right\}$

then write "DATA OUT OF RANGE"

Else write "IN PUT NEXT DATA SET"

What will happen if (i) X = 3 , Y= 2 and Z = 15

(ii) X = 6 , Y= 7 and Z = 10

(iii) X = 4 , Y= 2 and Z = 10

(iv) X = 1, Y= 2 and Z = 8

1.5 Algebra of propositions:

t – all values T f – all values F

1. **Idempotent laws**
   (a) $\boldsymbol{p} \vee \boldsymbol{p} \equiv \boldsymbol{p}$ (b) $\boldsymbol{p} \wedge \boldsymbol{p} \equiv \boldsymbol{p}$

2. **Associative laws**
   (a) $(\boldsymbol{p} \vee \boldsymbol{q}) \vee \boldsymbol{r} \equiv \boldsymbol{p} \vee (\boldsymbol{q} \vee \boldsymbol{r})$ **(b)** $(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge \boldsymbol{r} \equiv \boldsymbol{p} \wedge (\boldsymbol{q} \wedge \boldsymbol{r})$

3. **Commutative laws**
   (a) $\boldsymbol{p} \vee \boldsymbol{q} \equiv \boldsymbol{q} \vee \boldsymbol{p}$ **(b)** $\boldsymbol{p} \wedge \boldsymbol{q} \equiv \boldsymbol{q} \wedge \boldsymbol{p}$

4. **Distributive laws**
   (a) $\boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \equiv (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$
   (b) $\boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) \equiv (\boldsymbol{p} \wedge \boldsymbol{q}) \vee (\boldsymbol{p} \wedge \boldsymbol{r})$

5. **De Morgan's laws**
   (a) $\sim (\boldsymbol{p} \vee \boldsymbol{q}) \equiv \sim \boldsymbol{p} \wedge \sim \boldsymbol{q}$ **(b)** $\sim (\boldsymbol{p} \wedge \boldsymbol{q}) \equiv \sim \boldsymbol{p} \vee \sim \boldsymbol{q}$

6. **Identity laws**
   (a) $\boldsymbol{p} \vee \boldsymbol{f} \equiv \boldsymbol{p}$ **(b)** $\boldsymbol{p} \wedge \boldsymbol{t} \equiv \boldsymbol{p}$

7. **Identity laws**
   (b) $\boldsymbol{p} \vee \boldsymbol{t} \equiv \boldsymbol{t}$ **(b)** $\boldsymbol{p} \wedge \boldsymbol{f} \equiv \boldsymbol{f}$

8. **Complement laws**
   (a) $\boldsymbol{p} \vee \sim \boldsymbol{p} \equiv \boldsymbol{t}$ **(b)** $\boldsymbol{p} \wedge \sim \boldsymbol{p} \equiv \boldsymbol{f}$

9. Complement laws

(a) $\sim(\sim p) \equiv p$ (b) $(\sim t) \equiv f$ $\sim f \equiv t$

1.6 Conditional Statements

1.6.1 Conditional

Let p and q be propositions. The implication p$\Rightarrow$ q is the proposition that is false only when p is true and q is false; otherwise it is true. p is called the hypothesis and q is called the conclusion. The connective $\Rightarrow$ is called the conditional connective.

Example 1.6.1

Construct the truth table of the implication p$\Rightarrow$ q

Solution:

The truth table is

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1.6.2

Show that $p \Rightarrow q \equiv \sim p \vee q$

It follows from the previous exercise that the proposition p $\Rightarrow$ q is always true if the hypothesis p is false, regardless of the truth value of q: We say that p $\Rightarrow$ q is true by default or vacuously true. In terms of words the proposition p $\Rightarrow$ q also reads:

(a) if p then q

(b) p implies q

(c) p is a sufficient condition for q

(d) q is a necessary condition for p

Example 1.6.3

Use the if-then form to rewrite the statement "*I am on time for work* **if** I catch the 8:05 bus."

Example 1. 6. 4

a. Show that $\sim(p \Rightarrow q) \equiv p \vee \sim q$

b. Find the negation of the statement " If my car in the repair shop, then I cannot go to class."

1.6.2 Converse, Inverse and Contra positive:

Converse of proposition $p \Rightarrow q$ **is** $q \Rightarrow p$

Inverse of proposition $p \Rightarrow q$ **is** $\sim p \Rightarrow \sim q$

Contra positive of proposition $p \Rightarrow q$ **is** $\sim q \Rightarrow \sim p$

		Conditional	**Converse**	**Inverse**	**Contra positive**
p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Example 1.6.5

Find the converse, opposite, and the contra positive of the implication: " If today is Thursday, then I have a test today."

Example 1.6.6

Show that $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$:

Example 1.6.7

Using truth tables show the following:

(a) $\mathrm{p} \Rightarrow \mathrm{q} \neq \mathrm{q} \Rightarrow \mathrm{p}$

(b) $p \Rightarrow \mathrm{q} \neq \sim \mathrm{p} \Rightarrow \sim \mathrm{q}$

The biconditional proposition of p and q, denoted by p $\Leftrightarrow$ q, is the prepositional function that is true when both p and q have the same truth values and false if p and q have opposite truth values. Also reads, ”p if and only if q” or ”p is a necessary and sufficient condition for q.”

Example 1.6.8

Construct the truth table for p $\Leftrightarrow$ q

Example 1.6.9

Show that the biconditional proposition of p and q is logically equivalent to the conjunction of the conditional propositions p $\Rightarrow$ q and q $\Rightarrow$ p:

In propositional functions that involve the connectives ~; $\wedge$;_ $\vee$; and $\Rightarrow$ the order of operations is that ~ is performed first and $\Rightarrow$ is performed last.

The order of operations for the five logical connectives is as follows:

~

$\wedge$, $\vee$ in any order

$\Rightarrow$, $\Leftrightarrow$ in any order.

connective	symbol
and	$\wedge$
or	$\vee$
not	~
if then	$\Rightarrow$
if and only if	$\Leftrightarrow$

Exercises:

Q1. Let p be " He is old" and q be " he is clever". Write in symbolic form:

(i) He is clear and old
(ii) He is neither old nor clever
(iii) He is old but not clever
(iv) It is not true that he is young or not clever

Q2. Construct truth tables for the following:

(i) $\big((p \Rightarrow q) \wedge p\big) \Rightarrow q$ (ii) $p \Leftrightarrow (q \Rightarrow r)$ (ii) $(p \Leftrightarrow q) \Leftrightarrow r$

Q3. Show the following pair of statements are equivalent

(i) $\big((n = 6)\ OR\ (a > 4)\big)\ AND\ (x = 1)$; $\big((n = 6)\ AND\ (x = 1)\big)\ OR\ \big((a > 4)\ AND\ (x = 1)\big)$

(ii) $NOT\ \big((n = 6)\ AND\ (a \le 4)\ \big)$; $\big((n <> 6)\ OR\ (a > 4)\big)$

Q4. Write down the converse, inverse and contra positive to the following statements

(i) Every geometric figure with four right angles is a square.
(ii) All engineers have practical skills are good at mathematics.

Q5. Using truth tables find whether the following are tautologies or contradictions or neither

(i) $\sim(p \wedge q) \vee (\sim p \vee \sim q)$ (iii) $(\sim p \vee q) \wedge (p \vee \sim q)$

(ii) $\sim(p \wedge q) \vee \big(\sim(\sim p \wedge q)\big)$ (iv) $(p \wedge \sim q) \wedge (\sim p \vee q)$

1.7 Propositions and Quantifiers

1.7.1 Predicate:

Statements such as "$x > 3$" are often found in mathematical assertions and in computer programs. These statements are not propositions when the variables are not specified. However, one can produce propositions from such statements. A **predicate** is an expression involving one or more variables defined on some domain, called the **domain of discourse or Universe of discourse.** Substitution of a particular value for the variable(s) produces a proposition which is either true or false. Or other wards the generalization of propositions is called as propositional function or predicates. Or prepositions which contain variables. For instance,

$P(n)$: n is prime is a predicate on the natural numbers. Observe that $P(1)$ is false, $P(2)$ is true. In the expression *n* is called a free variable. As *n* varies the truth value of P(*n*) varies as well. The set of true values of a predicate $P(n)$ is called the truth set and will be denoted by T_P :

Example 1.7.1: $P(x)$: $x + 3 > 2$ is the predicate. It has no true or false value until the variable bound

Example 1.7.2:

$P(x)$: $x + 3 > 2 \quad x \in \Re$

Let $Q(x, y, z)$: $x + y = z$, it consist three **free variables** x, y *and* z.

$Q(1,2, z)$: $1 + 2 = z$ Here only z is the free variable other two are bond to values $x = 1$ and $y = 2$.

1.7.2 Quantifiers:

Quantifiers provide a notation that allows quantifying (count) how many objects in the universal of discourse satisfy a given predicate.

Example 1.7.3:

a. All positive integers $P(x)$: $x > 0$ is true
b. There exist x in real numbers such that $P(x)$: $x > 0$

Universal Quantification

For all values of x, $P(x)$is true or simply "for all x, $P(x)$" is called as universal quantification and denoted by $\forall\ x\ P(x)$

Example 1.7.4:

Let the universal of discourse of x be parking spaces at Majestic City (MC). Let $P(x)$ be the predicate "x is full" Then $\forall\ x\ P(x)$

- “ all parking spaces at MC are full”
- “ Every parking space at MC is full”
- “For each parking space at MC , that space is full”

Example 1.7.5:

Let $P(x)$: $|x| \geq 0$. Let universal disclouse of x is real number set. Then $\forall\ x\ P(x)$ means:

- “ mod value of all real numbers are positive”
- “ mod value of every real number is positive”

Existential Quantification

$P(x)$ is true for some x in the universal discourse. It is denoted by $\exists x\ \ P(x)$ and called as there exist x such that $P(x)$ is true.

Example 1.7.8:

Let $P(x) : x * 3 > 10 \quad x \in \mathcal{R}$

When x= 6.9 the statement is true, therefore we can say $\exists x\ \ P(x)$ (There exist x such that $P(x)$ is true).

Example 1.7.9:

Let the universal of discourse of x be parking spaces at Majestic City (MC). Let $P(x)$ be the predicate “x is full” Then $\exists\ x\ P(x)$ mans

- “ Some of parking spaces at MC are full”
- “ there is a parking space at MC that is full”
- “At least one parking space at MC is full”

Unique Existential Quantifier

The predicate is true for one and only one x in the universe of discourse. It is denoted by $\exists!\, P(x)$ and called as

- There is a unique x such that $P(x)$.
- There is one and only one x such that $P(x)$.
- One can find only one x such that $P(x)$.

Example 7.4.6

Consider predicate $P(x)$: $x \times 5 = 1\ x \in\ \mathcal{R}$, then we can say $\exists!\ x \in\ \mathcal{R}\ P(x)$.

Example 7.4.7

Let $P(x)$ denote the statement ”$x\ >\ 3$” What is the truth value of the proposition $\exists \mathrm{x} \in \mathrm{R}, \mathrm{P(x)}$

The proposition $\forall \ x \in D, P(x) \Rightarrow Q(x)$ is called the **universal conditional proposition**. For example, the proposition $\forall x \in R$; if $x > 2$ then $x^2 > 4$ is a universal conditional proposition.

Example 7.4.8

Rewrite the proposition "if a real number is an integer then it is a rational number" as a universal conditional proposition.

Example 7.4.9:

$(\forall x)\ F(x)$ - "all x, $F(x)$ is true

$\sim(\forall x)\ F(x)$ - " There is at least one x , that does not have the property F(x)"

$(\exists x)\ (\sim F(x))$ - " There is at least one x , that does not have the property F(x)"

$(\exists x)\ (F(x))$ - " There is at least one x , that have the property F(x)"

$\sim(\exists x)\ (F(x))$ - " No at least one x , that have the property F(x)"

$(\forall x)\ (\sim F(x))$ - " No at least one x , that have the property F(x)"

Therefore

$$\sim(\exists x)\ (F(x)) \equiv (\forall x)\ (\sim F(x))$$

$$\sim(\forall x)\ F(x) \equiv (\exists x)\ (\sim F(x))$$

Note: The propositions are read from left to right and that the order of quantified variables is important.

Example 7.4.10:

Consider $P(x, y)$: $x + y = 7$ Universe of discourse $\mathbb{R}$.

$(\forall x)(\forall y\ )\ P(x, y)$ - "all x, all y $P(x, y)$ is true

$(\forall x)(\exists y\ )\ P(x, y)$ - "all x, there exist y such that $P(x, y)$ is true

$(\exists x)(\forall y\ )\ P(x, y)$ - "there exist x, all y $P(x, y)$ is true

$(\exists x)(\exists y\ )\ P(x, y)$ - "there exist x, there exist y $P(x, y)$ is true

$$(\exists x)(\exists y\ )\ P(x, y) \equiv (\exists y)(\exists x\ )\ P(x, y)$$

$$(\forall x)(\forall y\ )\ P(x, y) \equiv (\forall y)(\forall x\ )\ P(x, y)$$

1.8 BOOLEAN ALGEBRA

Definition 8.1:

A Boolean algebra is an ordered 6-tuple $(S, 0, 1, +, *, ')$ in which S is a set of elements, containing the two distinct elements 0 and 1, + and $ are binary operations on S and ' is a unary operation on S such that the following properties hold.

1. **Commutative laws**

   $x + y = y + x \qquad x * y = y * x$

2. **Associative laws**

   $(x + y) + z = x + (y + z) \qquad (x * y) * z = x * (y * z)$

3. **Distributive laws**

   $x * (y + z) = x * y + x * z \qquad x + (y * z) = (x + y) * (x + z)$

4. **Identity laws**

   $x + 0 = x \qquad x * 1 = x$

5. **Complement laws**

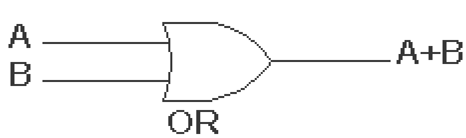
   $x + x' = 1 \qquad x * x' = 0$

1.9 Electronic Gates

Boolean functions may be practically implemented by using electronic gates. The following points are important to understand.

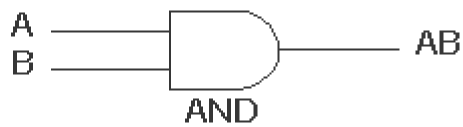
- Electronic gates require a power supply.
- Gate **INPUTS** are driven by voltages having two nominal values, e.g. 0V and 5V representing logic 0 and logic 1 respectively.
- The **OUTPUT** of a gate provides two nominal values of voltage only, e.g. 0V and 5V representing logic 0 and logic 1 respectively. In general, there is only one output to a logic gate except in some special cases.
- There is always a time delay between an input being applied and the output responding.

OR gate



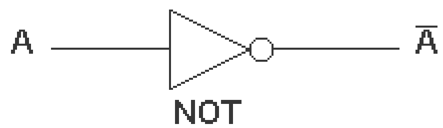
2 Input OR gate		
A	B	A+B
0	0	0
0	1	1
1	0	1
1	1	1

AND gate



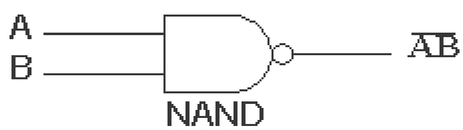
2 Input AND gate		
A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

NOT gate



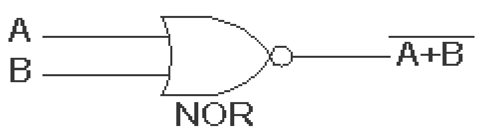
NOT gate	
A	$\overline{A}$
0	1
1	0

NAND gate



2 Input NAND gate		
A	B	$\overline{A.B}$
0	0	1
0	1	1
1	0	1
1	1	0

NOR gate



2 Input NOR gate		
A	B	$\overline{A+B}$
0	0	1
0	1	0
1	0	0
1	1	0

Example 1:

$x_1x_2' + x_3$ construct logic networks and truth table.

Example 2:

$(x'y)'$ construct logic networks and truth table.

Example 3:

$x'y + z'w + y'w$ construct logic networks and truth table.

CHAPTER 2

Mathematical Proof

2.1 Introduction:

A mathematical proof is probably that of sequence of steps, where each step follows logically from an earlier part of the proof and where the last line is the statement being proved. The proof must demonstrate that a statement is true in all cases, without a single exception. Making judgments about facts on the basis of observation is known as **inductive reasoning**. The type of reasoning where a conclusion is drawn by logical inference is called **deductive reasoning**. For mathematician, the latter is the only form of reasoning which is acceptable in a proof.

2.2 Axioms and Axiom Systems

The most important parts or gradients of mathematics proof are as follows:

a. Undefined terms
b. Axioms
c. Definitions
d. Theorems
e. Proofs

Undefined terms:

Some of the terms are not properly defined but we used for proofs. As a example consider definition of sets in mathematics set defined as “well defined group or collection” but the terms group or collection not defined in mathematics.

Axioms:

Axioms are statements which not proved but by experience accepted as correct. In general an accepted statement or proposition regarded as being self evidently true.

As example “Distinct parallel lines never meet”

Which is one of the foundations Euclidean geometry, but which is regarded as false in Riemannian geometry.

Definition: A definition is a passage describing the meaning of a word or phrase.

Example 1: The definition of a prime number “An integer greater than one that is not divisible by any positive integer other that 1 and itself”

A **proposition** is a true statement that you intend to prove. Theorems, lemmas and corollaries are all examples of propositions.

Theorem: It is a major result which derives from axioms, and undefined terms**. Lemma** is a component proposition of a theorem. Theorem sometimes consist series of lemmas. **Corollaries** are the results come immediately after the theorems.

Proof: In mathematics proof is a result based on some definite reasoning. Proof must demonstrate the statement is true for all cases. Proof is based on logical arguments.

2.3 Techniques of Proof:

2.3.1 Direct Method

A direct proof is one that follows from definitions, axioms. Facts previously learned help many a time when making direct proof. Many mathematical conjectures can be expressed in the form $\rightarrow Q$. That is as a conditional proposition. Their proof therefore consists of showing that

$$(A_1 \wedge A_2 \wedge A_3 \dots\dots \wedge A_n \wedge T_1 \wedge T_2 \dots..\wedge A_m) \longmapsto (P \Longrightarrow Q)$$

Where A_i , T_j are axioms and theorems.

Example 2:

Theorem: For every integer *n*, if *n* is even then n^2 is even. ($\forall\, n \in \mathbb{Z}$, $n\ is\ even \Longrightarrow n^2\ is\ even$)

Previous knowledge: An even number is one of the form $2k$ where *k* is an integer.

Example 3: Prove that if n is an integer $n^3 - n$ is always divisible by 6.

Example 4:

What wrong with the following proof:

"For x a real number, if $x = 2$ then $x = 1$ "

$$x = 2$$

$$\Longrightarrow \quad x - 1 = 1$$

$$\Longrightarrow \quad (x-1)^2 = 1$$

$$(x-1)^2 = x - 1$$

$$\Longrightarrow x^2 - 2x + 1 = x - 1$$

$$\Longrightarrow x^2 - 2x + 1 = x - 1$$

$$\Longrightarrow \quad x^2 - 2x = x - 2$$

$$\Longrightarrow \quad \frac{x(x-2)}{x-2} = \frac{x-2}{x-2}$$

$$\Longrightarrow \quad x = 1$$

2.3.2 The contra positive Method

When the proof is a conditional proposition we can use the use the contra positive equivalence to prove the theorem. That is inserted of proving $P \Longrightarrow Q$ we show that$\sim Q \Longrightarrow \sim P$. Then by the equivalence relation $P \Longrightarrow Q \equiv \sim Q \Longrightarrow \sim P$ we concluded that $P \Longrightarrow Q$.

Example 5:

Prove that, for every integer, n if $n^2$is even then n is even n

Let $P(n)$: $n^2$is even $Q(n)$: n is even

We have to prove $\forall\, n\ \in \mathbb{N}\quad P(n)\ \Longrightarrow Q(n)$

We prove that $\forall\, n\ \in \mathbb{N}\quad \sim Q(n)\ \Longrightarrow \sim P(n)$

$\sim Q(n)$ means n is odd then $n = 2k + 1$

Proof of a biconditional Proposition

To prove a biconditional proposition $P \Leftrightarrow Q$, we usually appeal to the logical equivalence of $P \Leftrightarrow Q$ and $(P \Longrightarrow Q) \wedge\ (Q \Longrightarrow P)$ .Commonly therefore the proof of a biconditional involves two distinct parts, one proving the result $(P \Longrightarrow Q)$ and other proving $(Q \Longrightarrow P)$.

Example 6: Prove that, for any integer *x* and *y* the product *xy* is even if and only if x is even or y is even.

2.3.3 Contradiction Method

In proof by contradiction (also known a reduction ad absurdum, Latin for "by reduction toward the absurd"). It is shown that if some statement were so, logical contradiction occurs, hence the statement must be not so. This method is perhaps the most prevalent of mathematical proofs. A famous example of a proof by contradiction shows that is an irrational number

Example 7: Show without using calculator that $6 - \sqrt{35} < \frac{1}{10}$

Example 8: Show that there are infinitely many prime numbers.

2.3.4 Construction Method

Another method of proof is particular useful for proving statements involving existential quantifiers (e. g. "there exists at least one….). This method works as follows:

"If such and such, then there exits an object such that so and so"

Example 8: Prove that if $a < b$ then there exits a real number c such that $a < c < b$.

2.3.5 Mathematical Induction Method

Let $\mathbb{N}$ denote the set of non–negative integers with normal ordering relation

$$a > b\ \Leftrightarrow a \neq b\ and\ \ a - b\ \in \mathbb{N}.$$

Thus $\mathbb{N} = \{0, 1, 2, \dots \dots \dots \dots .\}$

Theorem 1:

Well-ordering Axiom, Every non-empty subset of $\mathbb{N}$ contains a smallest element.

It is clear the axiom is true for every finite sub set of $\mathbb{N}$. However when the subsets considered are infinite there is no means to prove this axiom except by assuming an equivalence axiom. But neither can one prove (without new hypothesis) that this axiom is false; so we will run into no problems down the line by adopting this axiom from the start as a property of the set of natural numbers.

An important consequence of the Well-Ordering Axiom is the method of proof known as mathematical induction.

Principle of Mathematical Induction

Let $S(n)$ be a proposition concerning a positive integer, if

(a) $S(1)$ is true, and
(b) for every $k \geq 1$, the truth of $S(k)$ implies the truth of $S(k+1)$,
then $S(n)$ is true for all positive integer *n*.

Proof: Let

$$P = \{n \in \mathbb{N} |\ S(n) \text{ is false}\}$$

To prove the theorem, we need to show that P is empty. We shall use proof by contradiction to do this.

Suppose P is non-empty set. Then by the well-ordering Axiom P contains a smallest element, say d. Since $S(1)$ is true by hypothesis, and $S(d)$ is false, since $d \in P$, we must have $d \neq 1$. Consequently, $d > 1$; and so $d - 1 \in \mathbb{N}$.

Now $d - 1 < d$ and d is the smallest element of P, and so $d - 1 \notin P$. Therefore $S(d-1)$ is true.

However by property (ii) , with $k = d - 1$, implies that $S(d - 1 + 1) = S(d)$ is true statement. This is a contradiction since $d \in P$.

Therefore P is an empty set and the theorem is proved.

Example 9:

Prove that, for every positive integer n, the expression $2^{n+2} + 3^{2n+1}$ is divisible by 7.

There are various modifications which we can make to the inductive principle.

A. Suppose that we wish to prove that a proposition $S(n)$ is true for all integers greater than or equal to some fixed integer N. then steps of the method as follows
   (a) Prove that $S(N)$ is true
   (b) Prove that , for every $k \geq N$, the truth of $S(k)$ implies the truth of $S(k+1)$,
   then $S(n)$ is true for all positive integer *n*≥*N*.

Note: When we required to prove $S(n)$for all positive integers sometimes it can be simpler to begin the induction at $n = 0$ rather than $n = 1$.

Second Principle of Induction:

Let $S(n)$ be a proposition concerning a positive integer n. If

(a) $S(1)$ is true, and

(b) for every $k \geq 1$, the truth of $S(r)$ for all $r \leq k$implies the truth of $S(k+1)$, then $S(n)$ is true for all positive integer n.

Example 10: Prove that every positive integer greater than1 can be expressed as a product of prime numbers n.

2.3.6 Pigeonhole principle

The pigeonhole principle state that if n+1 pigeons fly to n holes, there must be a pigeonhole containing at least two pigeons. This is apparently trivial principle is very powerful. As example group of 13 people, there are always two have their birthday on the same month.

Example 11:

Let A be any set of twenty integers chosen from the arithmetic progression 1,4,….100. Prove that there must be two distinct integers in A whose sum is 104.

Example 12:

Given any 9 integers whose prime factors lie in the set $\{3, 7, 11\}$ prove that there must be two whose product is a square.