2. Fourier Series

Many phenomena that are studied in engineering are periodic in nature. For example the current and voltage in an alternating current circuit, the displacement, velocity and acceleration of the piston and many parameters in a vibrating system are all periodic. In order to solve such problems it is very often desirable that these functions are represented in infinite trigonometric series.

We would hope that such a modeling would give adequate representation over the whole cycle of periodicity rather than the local nature of the Taylor series representation.

Many wave phenomena are also periodic and it is well known, especially through acoustics that a wave can in general be decomposed or analyzed into several distinct waves of different frequencies.

Definition

A series of Sines and Cosines of the form $\frac{a_0}{2} + a_1 Cosx + a_2 Cos2x ... + b_1 Sinx + b_2 Sin2x + ...$

$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cosnx + \sum_{n=1}^{\infty} b_n Sinnx$ **,where** $a_0, a_1, ..., b_1, b_2, ...$ **are constants is called a Fourier series.**

We can show that a periodic function can be represented as a fourier series.

A periodic function f(x) whose period is 2π can be represented in the form

f(x) = $c_0 + c_1 Sin(x+\alpha) + c_2 Sin(2x + \alpha_2) + ...(1)$

approximately provided that

(i) f(x) is single- valued

(ii) f(x) is not infinite (i.e. infinite series is convergent).

Note:

A function f(x) is said to be periodic with period p if for all x,
f(x+p) = f(x), p is a positive constant. The least value of p satisfying the above relationship is called the **period of f(x).**

Second term of (1) can be written as

$c_1 Sin(x+\alpha_1) = c_1 CosxSin\alpha_1 + c_1 SinxCos\alpha_1$

$= a_1 Cosx + b_1 Sinx$

(where $a_1 = c_1 Sin\alpha_1, b_1 = c_1 Cos\alpha_1$)

Converting the remaining terms in a similar way f(x) becomes where c_0 has been written as $\frac{a_0}{2}$.(which would be shown later).

i.e. f(x) = $\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k Coskx + b_k Sinkx)$...............................(2)

i.e. A periodic function can be represented as a fourier series.

Consider the function Asin(nπx), where A and n are constants. This represents a wave where the amplitude of the wave is A, and the period will be 2/n, where n is the wavenumber. If we increase the wavenumber, then the function oscillated more rapidly, so sin(10x) has ten peaks and troughs between x=0 and x=2π whereas sin(x) only has one peak and one trough.

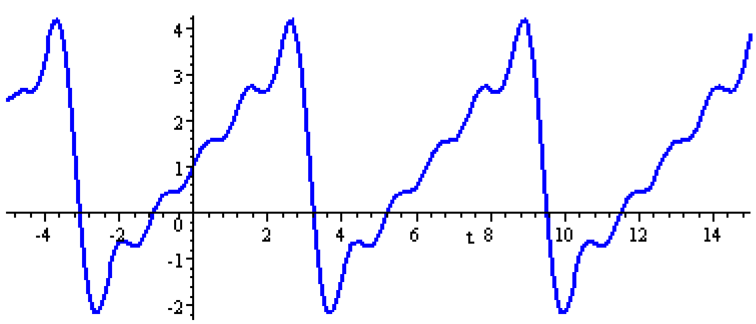
So the terms of a Fourier series represent waves of different periods (or wavelengths), with each successive term giving a more rapidly oscillating contribution than the previous.

The fundamental idea is that the large-scale features of any given periodic function can be roughly approximated by a sine or cosine wave, with suitably chosen amplitude. The shorter-scale features are more accurately captured by the subsequent terms in the Fourier series. In this way we can recreate the function as accurately as we choose, by taking more and more terms, thus including shorter and shorter scales.

Fourier Series is used in the analysis of signals in electronics.

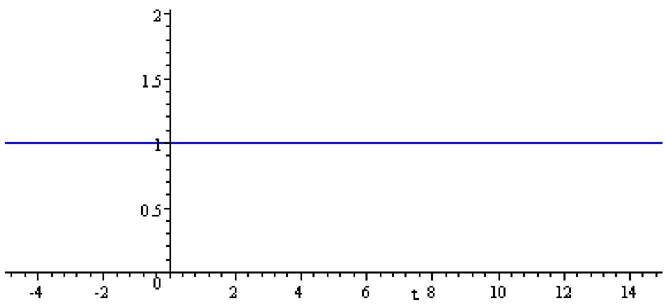
We will see functions like the following, which approximates a saw-tooth signal:

$$f(x) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t + \ldots$$

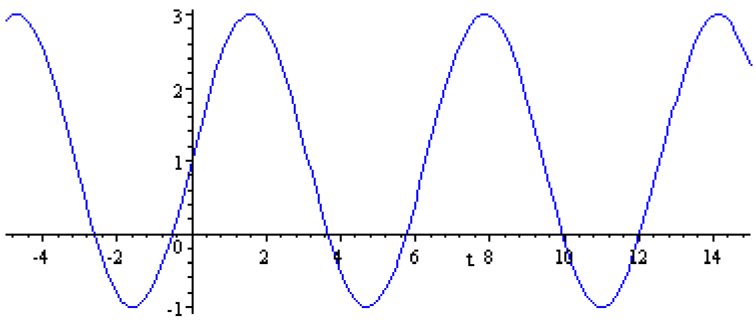


Taking one extra term in the series each time and drawing separate graphs, we have:

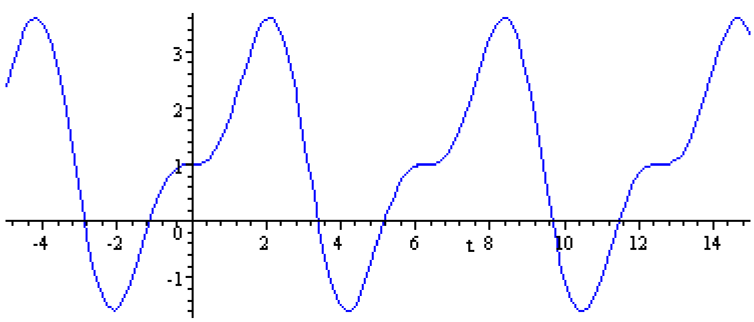
$f(t)$ = 1 (first term of the series):



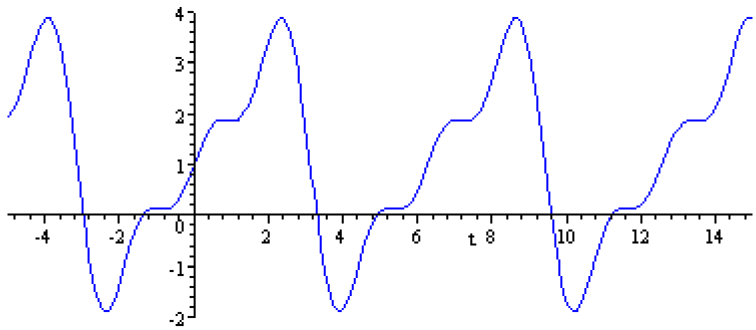
$f(t) = 1 + 2 \sin t$ (first 2 terms of the series):



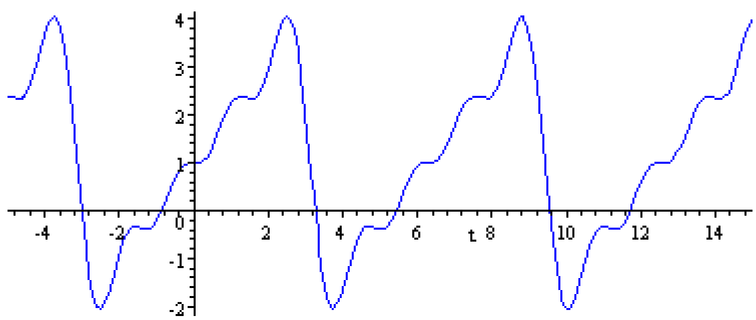
$f(t) = 1 + 2 \sin t - \sin 2t$ (first 3 terms of the series):



$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t$$



$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$$



$$f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t$$

We say that the infinite Fourier series **converges** to the saw tooth curve.

That is, if we take more and more terms, the graph will look more and more like a saw tooth. If we could take an **infinite** number of terms, the graph would look like a set of saw teeth...

Note:

(i) Whether the series converges or not will depend on the value of x chosen and the coefficients a_k and b_k.

(ii) If the series converges in any closed interval $[c, c+2\pi]$, the periodic nature of the Cosine and Sine functions guarantees convergence for all values of x.

(iii) We try to approximate a given function in the fourier series. For that it is necessary to know the values $a_0, a_1, \ldots\ldots, a_n$ and $b_1, b_2, \ldots\ldots$ which are called the fourier coefficients.

Definition:

If f(x) is defined on an interval $-\pi \leq$ x $\leq \pi$ with f(x)=f(x+2π). We define the fourier series of f(x) on [-π,π] as the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k Coskx + b_k Sinkx) \quad \text{------------} \quad \textbf{(3)}$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Coskxdx, k = 1,2......$$

$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinkxdx, k = 1,2.......$ **and fourier series of** $f(x)$ **is denoted by** $f_s(x)$.

In otherwords $f(x)$ **=** $f_s(x)$. **approximately.**

Proof is shown later.

That is, if the coefficients of the series (1) are determined in the case where the representation of a given function in the form (1) is valid then the series with these coefficients is called Fourier series of the function.

i.e. we are approximating a periodic function by an infinite trignometric series.

Integrals useful to calculate fourier coefficients:

(i) $\int_{-\pi}^{\pi} Cosnxdx = 0$ **and** $\int_{-\pi}^{\pi} Sinnxdx = 0$

(ii) $\int_{-\pi}^{\pi} CosmxCosnxdx = \frac{1}{2}\int_{-\pi}^{\pi}\{Cos(m+n)x + Cos(m-n)x\}dx$

$$= \begin{cases} \frac{1}{2}\int_{-\pi}^{\pi}(Cos2nx+1)dx = \pi, m = n \\ 0, m \neq n \end{cases}$$

(iii) $\int_{-\pi}^{\pi} Sinmx Sinnx dx = \frac{1}{2}\int_{-\pi}^{\pi}\{Cos(m-n)x - Cos(m+n)\}dx$

$$= \begin{cases} 0, m \neq n \\ \pi, m = n \end{cases}$$

(iv) $\int_{-\pi}^{\pi} Sinmx Cosnx = \frac{1}{2}\int_{-\pi}^{\pi}\{Sin(m+n)x + Sin(m-n)x\}dx = 0$

in all cases.

Same result holds for other limits provided they differ by 2π .

<u>Proof of equation (3)</u>

Consider,

f(x) = $\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k Coskx + b_k Sinkx)$ **…………………(1)**

Integrating both sides from $-\pi$ **to** π **,**

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\frac{1}{2}a_0 dx = \frac{1}{2}a_0.2\pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

To find a_1**, multiply by Cos x and integrate,**

$$\int_{-\pi}^{\pi} f(x)Cosxdx = \int_{-\pi}^{\pi} a_1 Cos^2 xdx = a_1\pi$$

$$a_1 = \int_{-\pi}^{\pi} f(x)Cosxdx$$

To find $a_2, a_3..........,a_n$ multiply (1) by Cos 2x, Cos 3x,…… and integrate between $-\pi$ to π and the general term is,

$$\int_{-\pi}^{\pi} f(x)Cosnxdx = \int_{-\pi}^{\pi} a_n Cos^2 nxdx = a_n\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cosnxdx$$

To find b_n **multiply by Sin nx and integrate,**

$$\int_{-\pi}^{\pi} f(x) Sinnxdx = \int_{-\pi}^{\pi} b_n Sin^2 nxdx = b_n \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sinnxdx$$

<u>Note:</u> a_n **and** b_n **may be written**

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) Cosnxdx$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) Sinnxdx$$

<u>Example</u>:

Develop f(x) in a fourier series in the interval $[-\pi,\pi]$ if

$$\mathbf{f(x)} = \begin{cases} 0 & ,-\pi \le x < 0 \\ 1 & ,0 \le x < \pi \end{cases}$$

Let $f_s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n Cosnx + b_n Sinnx)$

Then, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0dx + \int_{0}^{\pi} 1dx\right] = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Cosnxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0Cosnxdx + \frac{1}{\pi}\int_{0}^{\pi} Cosnxdx$$

$$= \frac{1}{\pi}\left[\frac{Sinnx}{n}\right]_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) Sinnxdx = \frac{1}{\pi}\int_{0}^{\pi} Sinnxdx = \frac{1}{\pi}\left[\frac{-Cosnx}{n}\right]_0^{\pi}$$

$$= \frac{1}{n\pi}(1 - Cosn\pi), n = 1,2,....$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

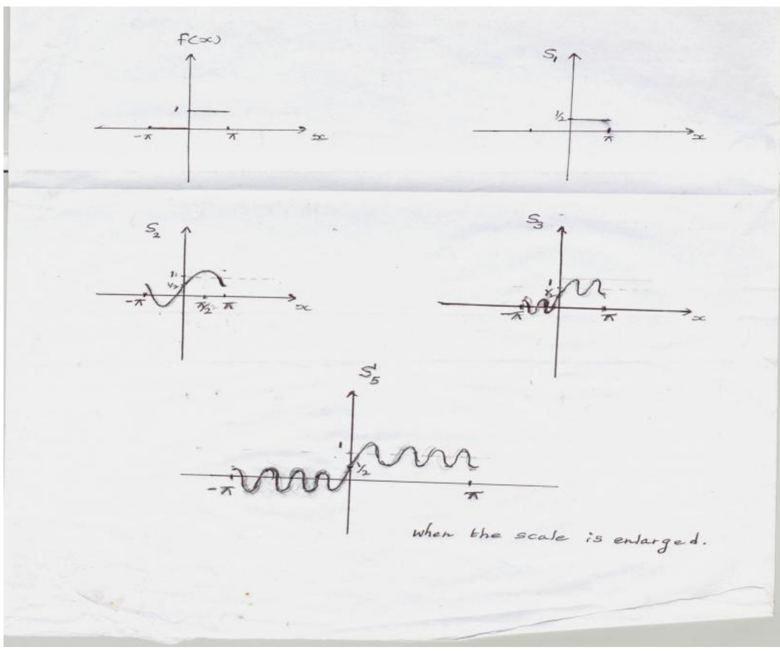
$b_1 = \frac{2}{\pi}$, $b_2 = 0,$ $b_3 = \frac{2}{3\pi}$, $b_4 = 0$

$$\therefore\ f_s(x) = \frac{1}{2} + \frac{2}{\pi}\left(Sinx + \frac{1}{3}Sin3x + \frac{1}{5}Sin5x + ... \right)$$

Note: The partial sums of the series are $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{2}{\pi}Sinx$,

$$s_3 = \frac{1}{2} + \frac{2}{\pi}(Sinx + \frac{1}{3}\sin 3x)$$

The graphs of f(x) and partial sums are shown below.



It is evident from the graphs that the accuracy with which S_n represents $f(x)$ increases with n.

Exercise 16

Find the fourier series for

(1) f(x) = x+2, $-\pi \le x \le \pi.$.

(2) f(x) = Cos $\frac{x}{2}, -\pi < x < \pi$

Odd and even functions

Definition: Function $f(x)$ is odd iff $f(-x) = -f(x)$ and
Function $f(x)$ is even iff $f(-x) = f(x)$.

Note: (i) Graph of an odd function is symmetrical about the origin whereas the graph of an even function is symmetrical about the y-axis.

(ii) (a) A product of two even or odd functions is even.
(b)A product of an even and odd functions is odd.
(c) For an even function $f(x)$

$$\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$$

(d) For an odd function $f(x)$

$$\int_{-\pi}^{\pi} f(x)dx = 0$$

(e) Fourier series of an even function is, $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

(f)Fourier series of an odd function is $\sum_{n=1}^{\infty} b_n \sin nx$

Examples: Find fourier series of the following functions.

(1) f(t) = $\begin{cases} -1 & , -\pi < t < 0 \\ 1 & , 0 < t < \pi \end{cases}$ (2) f(t) = $\begin{cases} -1 & , \quad -\pi < t < -\frac{\pi}{2} \\ 1 & , \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \\ -1 & , \quad \frac{\pi}{2} < t < \pi \end{cases}$

Dirichlet's theorem :

If f(x) is single valued and bounded periodic function which in any one period has a finite number of discontinuities and a finite number of maxima and minima , then

$S_p(x) = \frac{a_0}{2} + \sum_{n=1}^{p}(a_n \cos nx + b_n \sin nx)$ **converges to f(x) as** $p \to \infty$ **at values of x for which f(x) is continuous and to** $\frac{1}{2}\left[f(x+0) + f(x-0)\right]$ **(i.e. the average of the right – hand and left- hand limits of f(x)) at points of discontinuity.**

Exercise 17

(i) Given that $f(x) = x + x^2$ **for** $-\pi < x < \pi$ **with** $f(x+2\pi) = f(x)$. **Show that**

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

(ii) **Find fourier series for** $f(x)$**, if** $f(x) = \begin{cases} -\pi & ,-\pi < x < 0 \\ x & ,0 < x < \pi \end{cases}$.

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ... = \frac{\pi^2}{8}$. **Graph the fourier series.**

Half range series

We can often make use of the oddness or evenness of a function giving rise to Sine series or a Cosine series respectively. In the solution of some partial differential equations, the boundary conditions may restrict us to a series which contains only Sine terms. We shall therefore need to investigate how to manufacture an odd function or an even function, given a function which may be neither.

Suppose f(x) is defined on [0, π **] , and we want a Cosine series. We extend f(x) to a new function g(x) defined on [-**π,π**] as follows**

$$\textbf{g(x)} = \begin{cases} f(x) & ,0 \le x \le \pi \\ f(-x) & ,\pi \le x < 0 \end{cases}$$

Then g(x) is even in [-π,π**] . Hence** $g_s(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n Cosnx$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} g(x)dx$

$$a_n = \frac{2}{\pi}\int_0^{\pi} g(x)Cosnxdx$$

As g(x) = f(x) in (0, π), **therefore in this interval**

f $_s$ **(x) =** $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n Cosnx$ **…………..(1)**

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)Cosnxdx$$

This is called fourier <u>**half-range cosine series**</u> **for** $x \in (0,\pi)$**. It is denoted by** $f_{cs}(x)$.

Therefore, $f_s(x) = f_{cs}(x)$ **for** $x \in (0,\pi)$. **Where,** $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$,

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nxdx$ **.**

$g(x)$ is called even periodic extension of $f(x)$. Thus the series converges to $f(x)$ at all points on $(0,\pi)$ where $f(x)$ is continuous and at all points of discontinuity the series converges to the average of its left and right hand limits.

Suppose we want a Sine series. We can extend f(x) to an odd function h(x) on [- $\pi,\pi]$ **.**

$$h(x) = \begin{cases} f(x) & 0 \le x \le \pi \\ -f(-x) & ,-\pi \le x < 0 \end{cases}$$

$$h_s(x) = \sum_{n=1}^{\infty} b_n Sinnx$$

Here $b_n = \frac{2}{\pi}\int_0^{\pi} g(x)Sinnxdx$

Since h(x) = f(x) in $(0,\pi)$, therefore in this interval

$f_s(x)$ = $\sum_{n=1}^{\infty} b_n Sinnx$ **……………(2)**

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) Sinnxdx$

This is called fourier half-range sine series and it is denoted by $f_{ss}(x)$.

i.e. $f_s(x) = f_{ss}(x)$. $h(x)$ is called odd periodic extension of $f(x)$.

Thus the series converges to $f(x)$ at all points on $(0,\pi)$ where $f(x)$ is continuous and at all points of discontinuity the series converges to the average of its left and right hand limits.

Thus a function f(x) , whether even or odd or neither, defined over the interval $(0,\pi)$ **is capable of these two distinct expansions.**

Series (1) and (2) with coefficients defined above are called Fourier half range cosine series and sine series of f(x) respectively.

<u>Exercise 18</u>

1. Expand $f(x) = \sin x$ in a fourier cosine series in $0 < x < \pi$. Graph the fourier series of the even periodic extension of $f(x)$.

2. Represent the following by a fourier sine series in the given region and obtain the graph of the fourier series of the odd periodic extension of the function.

$$f(x) = \begin{cases} t & ,0 < t \le \frac{\pi}{2} \\ \frac{\pi}{2} & ,\frac{\pi}{2} < t < \pi \end{cases}$$

2.5 Fourier series of general period

So far only functions with period 2π have been considered. In practice it is often necessary to find a fourier series of $f(x)$ defined over the interval -L to L or 0 to 2L.

The fourier series of $f(x)$ **defined on (-L,L) with period 2L is,**

$$f_s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n Cos\ \frac{n\pi x}{L} + b_n Sin\ \frac{n\pi x}{L})$$

Where $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$ **,** $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\ Cos\ \frac{n\pi x}{L} dx$ **,** $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\ Cos\ \frac{n\pi x}{L} dx$

The fourier series of $f(x)$ defined on (0,2L) with period 2L is,

$$f_s(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\ \frac{n\pi x}{L} + b_n Sin\ \frac{n\pi x}{L}\right)$$

Where $a_0 = \frac{1}{L}\int_0^{2L} f(x)dx$ **,** $a_n = \frac{1}{L}\int_0^{2L} f(x)\ Cos\ \frac{n\pi x}{L}dx$ **,** $b_n = \frac{1}{L}\int_0^{2L} f(x)\ Cos\ \frac{n\pi x}{L}dx$

Exercise 19

1. Find the fourier series expansion of the following periodic function of period 1

$$f(x) = \begin{cases} \frac{1}{2} + x & , -\frac{1}{2} < x \le 0 \\ \frac{1}{2} - x & , 0 < x < \frac{1}{2} \end{cases}$$.

2. Expand $f(x) = Cosx,\ 0 < x < \frac{\pi}{2}$ in a fourier Sine series. Graph the fourier expansion of odd periodic extension of $f(x)$.

3. Find a fourier Cosine series for $f(x) = \begin{cases} x & , x < 1 \\ 2 & , x \ge 1 \end{cases}$ on (o, 2).Graph the fourier series of the even extension of the function function.