References:

(i) "Advanced Engineering Mathematics", K.A. Stroud, Dexter J. Booth
(ii) " Engineering Mathematics", H.K. Dass
(iii) " Higher Engineering Mathematics", Dr. B.S. Grewal

The method of assessment

This consists of the following exams with the given contributions to the total.

(1) Mid-term exam : 30% of the total
(Open book)
This consists of one hour paper with 15 multiple choice questions.

(2) Final Exam : 70% of the total

The paper consists of 6 questions and 5 questions should be answered.

1. Laplace transforms

In mathematics , " transform" usually refers to a device which changes one kind of function or equation into another kind. One attempts to design transforms which change problems that we do not know how to solve into problems which are to solve. This technique has proved very effective in solving differential equations.Many different transforms have been invented. In this section we study one of them, the Laplace transforms which are generally applied to complex electrical circuits and mechanical systems.

Definition 1

The Laplace transform $L\{f(t)\}$ of a function f(t) is defined to be

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$$

whenever this integral exists. The integration variable is t. Hence the integral defines a function of the new variable s.

We shall also customarily use t as the variable of our original function and s as the variable of its Laplace transform.

Definition 2

The inverse Laplace transform of F(s) is a function f(t) such that L{f(t)} = F(s). If we denote the operation of taking a Laplace transform by L, and of taking an inverse Laplace transform by L^{-1}, then

$$L\{f(t)\} = F(s) \quad \textbf{implies} \quad L^{-1}\{F(s)\} = f(t)$$

and conversely,

$$L^{-1}\{F(s)\} = f(t) \quad \textbf{implies} \quad L\{f(t)\} = F(s)$$

Exercise 1

Using the definition of Laplace transform show the following.

(i) L{t} = $\frac{1}{s^2}$, (s>0)

(ii) For s>0, L{Cos(at)} = $\frac{s}{s^2 + a^2}$

(iii) $L\{Sinat\} = \frac{a}{s^2 + a^2} \quad , s > 0$

(iv) $L\{e^{at}\} = \frac{1}{s-a} \quad , \quad s > a$

Note: It is shown by repeated integration that

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$ for any positive integer.

<u>Theorem 1</u> :

(i) L{f(t) $\pm$ g(t)} = L{f(t)} $\pm$ L{g(t)} whenever all these transforms exists.

Hence $L^{-1}\{L\{f(t)\} \pm L\{g(t)\}\} = f(t) \pm g(t)$

(ii) For any real number a ,L{a f(t)} = a L{f(t)} whenever both sides exists.

Hence $L^{-1}\{aL\{f(t)\}\} = a\ f(t)$ **.**

<u>Exercise 2</u>

Using above theorem find

(i) $L\{Sinhat\}$

(ii) $L\{4Sinh(3t) - 18e^{-5t}\}$

(iii) $L\{t^3 - 8t^2 + 1\}$

Laplace transforms of basic functions are given below.

$f(t)$	$L\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2!}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
Sin at	$\frac{a}{s^2 + a^2}$
Cos at	$\frac{s}{s^2 + a^2}$
(Above formulas are valid for $s > 0.$**)**	
e^{at}	$\frac{1}{s-a}$
Sinh at	$\frac{a}{s^2 - a^2}$
Cosh at	$\frac{s}{s^2 - a^2}$

(The above three formulas are valid for $s > a$ **).**

Exercise 3

Using the definition of inverse Laplace transforms obtain the following.

(i) $L^{-1}\left\{\frac{1}{s^5}\right\}$ (ii) $L^{-1}\left\{\frac{1}{s^2+64}\right\}$ (iii) $L^{-1}\left\{\frac{3s-2}{s^3(s^2+4)}\right\}$

(iv) $L^{-1}\{\frac{3s+5}{s^2+7}\}$ (v) $L^{-1}\{\frac{1}{(s-1)(s+2)(s+4)}\}$

Theorem 2 : **If a is any real number then**

$L\{e^{at} f(t)\} = F(s-a)$ where $F(s) = L\{f(t)\}$

i.e. $L\{e^{at} f(t)\} = L\{f(t)\}_{s \to s-a}$

This is known as first translation theorem or Shifting property.

Exercise 4

Find the following.

(i) $L\{e^{5t} t^3\}$ (ii) $L\{e^{-2t} Cos 4t\}$ (iii) $L^{-1}\{\frac{s+1}{s^2(s+2)^3}\}$

(iv) $L\{e^{at} Cos\, bt\}$ (v) $L\{e^{at} Sin\, bt\}$ (vi) $L\{e^{at} Cosh\, bt\}$

(vii) $L\{e^{at} Sinh\, bt\}$ (viii) $L\{Cosh\, at Cos\, bt\}$

Laplace Transform of a Derivative

Our goal is to use the Laplace transforms to solve certain kind of differential equations. For that we need to evaluate quantities such as $L\{\frac{dy}{dt}\}$ and $L\{\frac{d^2y}{dt^2}\}$.

if f' is continuous then , for $t \geq 0$

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Exercise 5

Prove above two results.

Note:

In general ,

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ....f^{(n-1)}(0)$$

where F(s) = L{f(t)}.

Solving differential equations using Laplace transforms

In order to solve differential equations Laplace transforms of derivatives are used.

Exercise 6

Solve the following.

(i) $\frac{dy}{dt} - 3y = e^{2t}$ subject to y(0) = 1.

(ii) $y'' - 6y' + 9y = t^2 e^{3t}$ subject to y(0) = 2 , $y'(0) = 6$

(iii) $y'' + 4y' + 6y - 1 = e^{-t}$ subject to y(0) =0, $y'(0) = 0$

(iv) $x'' + 16x = Cos4t$ subject to X(0) =0 , $x'(0) = 1$

Solving Simultaneous differential equations using Laplace transforms

The Laplace transforms reduce a system of linear equations with constant coefficients to a set of simultaneous algebraic equations in the transformed functions.

Exercise 7

Solve the following.

(i) $2x' + y' - y = t$

$x' + y' = t^2$ subject to x(0) = 1, y(0)= 0

(ii)Mechanical system with two degree of freedom satisfies the equations mentioned below.

$\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4$, $2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0$. Use laplace transforms to determine x and y at any instant given that $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at t=0.

Some important theorems in laplace transforms

Theorem 3: $L\{\int_0^t f(u)du\} = \frac{1}{s}F(s)$ where L{f(t)} = F(s).

Theorem 4: If $L\{f(t)\} = F(s)$ then $L\{tf(t)\} = -\frac{d}{ds}(F(s))$.

Similarly,

$$L\{t^2 f(t)\} = \frac{d^2}{ds^2}\{F(s)\}$$

In general $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$

Exercise 8

Find the following.

(i) L{t Cos at} (ii) L{ t^2 Sint} (iii) $L^{-1}\left\{\frac{a}{s(s^2+a^2)}\right\}$ hence find $L^{-1}\left\{\frac{a}{s^2(s^2+a^2)}\right\}$

(iv) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Theorem 5: $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$ where $F(s) = L\{f(t)\}$ then

$$\left\{\frac{f(t)}{t}\right\} = L^{-1}\{\int_s^\infty F(s)ds\}.$$

Exercise 9

Find laplace transform of (i) $\int_0^t \frac{Sin\,2u}{u}du$ (ii) $\frac{e^{-2t}Sin\,3t}{t}$

Theorem 6: If f(t) is periodic with period T>0, then

$$L\{f(t)\} = \left\{\frac{\int_0^T e^{-st}f(t)dt}{(1-e^{-sT})}\right\}$$

Exercise 10

Find $L\{|Sin\, t|\}$.

Theorem 7: If $U_a(t)$ is defined as follows ,

$$U_a(t)=\begin{cases}0 & ,t<a\\ 1 & ,t\geq a\end{cases}$$

then $L\{U_a(t)\}=\dfrac{e^{-as}}{s}\Rightarrow \quad L^{-1}\left\{\dfrac{e^{-as}}{s}\right\}=U_a(t)$.

Exercise 11

(i)Obtain the following

(a) If f(t) = k { $U_a(t)-U_b(t)$ } then $L\{f(t)\}$.

(b) If $g(t) = U(t)-2U_a(t)-2U_{2a}(t)-2U_{3a}(t)+.........$ then . $L\{g(t)\}$

(iii) Obtain the graph of $f(t) = U_2(t)t^2$.

(iv)For t>0 sketch the graphs of the following.

(a) $f(t)=U_a(t)e^{-t}$

(b) $f(t)=U_a(t)e^{-(t-a)}$

(c) $f(t)=e^{-t}\{U_1(t)-U_2(t)\}$

(v) Express the following function in terms of unit step functions.

$$F(t) = \begin{cases}1 & , \quad 0\leq t<a\\ 2 & , \quad a\leq t<2a\\ 3 & , \quad 2a\leq t<3a\end{cases}$$

Hence find laplace of F(t).

Theorem 8: Second Shifting Property

If $L\{f(t)\}=F(s), \quad L\{U_a(t)f(t-a)\}=e^{-as}F(s)$.

$$= e^{-sa}\int_0^{\infty} e^{-su} f(u)du = e^{-sa}F(s).$$

Ctd…

Exercise 12:

(1) A function f(t) is defined by

$$f(t) = 4 \quad ,0 < t < 2$$
$$= 2t - 3 \quad ,2 < t$$

Sketch the graph of the function and determine its Laplace transform.

(2) Express the following in terms of unit step function.

(a) $f(t) = \begin{cases} 8 & ,t < 2 \\ 6 & ,t > 2 \end{cases}$

(b) $f(t) = \begin{cases} t - 1 & ,1 < t < 2 \\ 3 - t & ,2 < t < 3 \end{cases}$

(3) A resistance R_0 in series with inductance L_0 is connected with E(t) . The current i is given
by,

$$L_0 \frac{di}{dt} + R_0 i = E(t)$$

If the switch is connected at t=0 and disconnected at t=a , find the current i(t) in terms of t.

4) If $f(t) = L^{-1}\left\{\frac{(1 - e^{-2s})(1 + e^{-4s})}{s^2}\right\}$ determine f(t) and sketch the graph of the function.

Theorem 9: Convolution Theorem

If $L^{-1}\{F(s)\} = f(t)$ **and** $L^{-1}\{G(s)\} = g(t)$ **then**

$$L^{-1}\{F(s).G(s)\} = \int_0^t f(u)g(t-u)du$$

Exercise 14:

(1) Applying convolution theorem solve the following initial value problem.
$y'' + y = Sin3t$, $y(0) = 0$ and $y'(0) = 0.$

(2)Apply convolution theorem to find the following.

(a) $L^{-1}\left\{ \frac{s}{(s^2 + a^2)^2}) \right\}$ (ii) $L^{-1}\left\{ \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right\}$