Lecture 1
CALCULUS

1. Introduction

1.1 What is vector calculus?

Vector calculus is how to define and measure the variation of temperature, fluid velocity, force, magnetic flux etc. over all three dimensions of space. In the real 3D engineering world, one wants to know things like the stress and strain inside a structure, the vorticity of the air flow over a wing, or the induced electromagnetic field around an aerial. For such questions, it is simply not good enough to deal with $\frac{dy}{dx}$ and $\int f(x)dx$. We must instead know how to integrate and differentiate *vector* quantities with three components (in directions i, j and k) which depend on three co-ordinates x, y, z.
Vector calculus provides the necessary mathematical notation and techniques for dealing with such issues. First, let's recall what we mean by vectors and calculus in isolation.

1.2 Vectors (revision)

- Notation: $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k} = \langle v_1, v_2, v_3 \rangle$
- length: $|\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ unit vector: v = $\underline{v} = \frac{v}{|v|}$

Position vector: $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \langle x, y, z \rangle$

The Dot Product :
We define the *dot product* of two vectors

v = a**i** + b**j** and **w** = c**i** + d**j**

to be **v** · **w** = ac + bd

Notice that the dot product of two vectors is a number and not a vector. For 3 dimensional vectors, we define the dot product similarly:

Dot Product in R^3
If **v** = a**i** + b**j** + c**k** and **w** = d**i** + e**j** + f**k**
then
v · **w** = ad + be + cf

Exercise:
Find the dot product of 2i + j - k and i + 2j

The Angle Between Two Vectors

We define the angle theta between two vectors v and w by the formula

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

so that

$$\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$$

Two vectors are called *orthogonal* if their angle is a right angle.

We see that angles are orthogonal if and only if $\mathbf{v}\cdot\mathbf{w} = 0$.

Example:
Find the angle between $\mathbf{v} = 2i + 3j + k$ and $\mathbf{w} = 4i + j + 2k$

The Cross Product Between Two Vectors
Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ be vectors then we define the *cross product* **v** x **w** by the determinant of the matrix:

$$\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{pmatrix}$$

Example:

Find the cross product **u** x **v** if $u = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $v = 4\mathbf{j} + 5\mathbf{k}$

Exercises:
Find **u** x **v** when
A. $\mathbf{u} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
B. $\mathbf{u} = 2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Notice that since switching the order of two rows of a determinant changes the sign of the determinant, we have **u** x **v** = **-v** x **u**

Parallelepipeds
To find the volume of the parallelepiped spanned by three vectors **u**, **v**, and **w**, we find the scalar triple product:

Volume = **u**. (**v** x **w**)

Example:
Find the volume of the parallelepiped spanned by the vectors
$\mathbf{u} = \langle 1,0,2\rangle$ $\mathbf{v} = \langle 0,2,3\rangle$ $\mathbf{w} = \langle 0,1,3\rangle$

NOTE:

- Scalar triple product $\underline{a}\cdot(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\cdot\underline{c} = -(\underline{a}\times\underline{c})\cdot\underline{b}$
- vector triple product $\underline{a}\times(\underline{b}\times\underline{c})$

$$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c} \neq (\underline{a}\times\underline{b})\times\underline{c} = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{b}\cdot\underline{c})\underline{a}$$

1.3 Scalar field, vector field and Scalar functions

1. A scalar function (of one variable) $f(x)$ or $f(t)$ is a formula that takes a scalar and returns a scalar. It might be used to describe the spatial variation of temperature T(x) along a one-dimensional bar heated at one end, or the time variation of the DC current $i(t)$ across a certain component in an electrical circuit.

2. **A scalar field** ϕ is a scalar quantity defined over a region of space. It takes a vector (of positions) and returns a scalar.

$\phi = f(x, y, z) = f(r)$ (or $f(x, y)$ in 2D).

Eg: The variation of temperature T (x, y, z) in this room using Cartesian co-ordinates. We might also think of the variation of density or charge density $\rho(x, y, z)$ inside a solid object.

3. A **vector field** $\underline{v}(x, y, z)$ is a vector-valued quantity defined over a region of space. It is defined by a function that takes a vector (of positions) and returns a vector

$$\underline{v} = v_1(x, y, z)\underline{i} + v_2(x, y, z)\underline{j} + v_3(x, y, z)\underline{k}$$

$$\underline{v} = v_1(x, y)\underline{i} + v_2(x, y)\underline{j} \quad \text{(in 2D)}$$

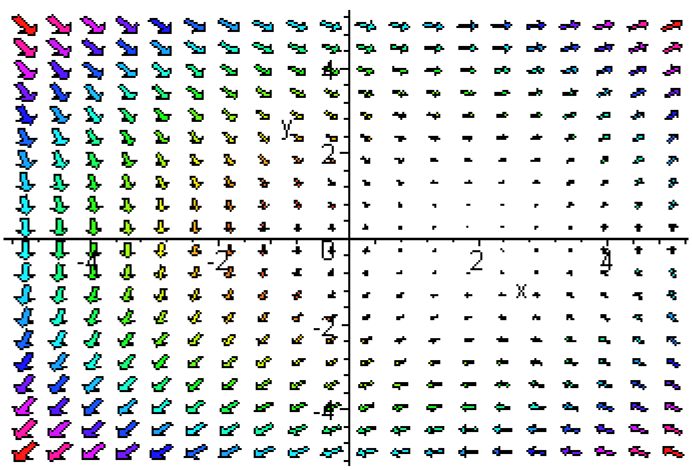
Eg: The spatial variation of fluid velocity $\underline{v}(x, y, z)$ in a steady flow, or current $I(x, y, z)$ flowing in a conductor.

Example:

An important vector field that we have already encountered is the *gradient vector field.* Let f(x,y) be a differentiable function then the function that take a point (x_0,y_0) to $\text{grad}f(x_0,y_0)$ is a vector field since the gradient of a function at a point is a vector.

For example, if f(x,y) = 0.1xy - 0.2y then gradf(x,y) = 0.1y**i** + (0.1x - 0.2)**j**

The sketch of the gradient is pictured below.



1.4 Vector functions

A **vector function** (of one variable) $v\,(x)$ or $v\,(t)$ takes a scalar and returns a vector: $\underline{v} \quad = v_1(t)\underline{i} + v_2(t)\underline{j} + v_3(t)\underline{k}$. Such functions might be used to describe the motion of a particle whose position vector ' is $\underline{r}\ (t)$ at time *t;* or the external forces $\underline{F}\ (x)$ acting at distance x along a 1-dimensional case.

Differentiation and integration of vector functions are easy! One simply differences or integrates the components separately.

$$\frac{d}{dt}\underline{v} \quad = \frac{d}{dt}v_1(t)\underline{i} + \frac{d}{dt}v_2(t)\underline{j} + \frac{d}{dt}v_3(t)\underline{k}$$

Example:

A particle moves on a circle of radius 1, such that its position vector is given by

$\underline{r}(t) \quad = \sin t\underline{i} + \cos t\,\underline{j}$

Calculate its velocity and acceleration. Show that the velocity and acceleration are orthogonal.

Example:

Sketch the graph of each of the following vector functions.

$$\vec{r}(t) = \langle t, 1 \rangle$$

- **Rules of differentiation**

. Here $\underline{u} = \underline{u}(t)$, $\underline{v} = \underline{v}(t)$ and c is a constant

1. $(\underline{u} + \underline{v})' \quad = \quad \underline{u}' + \underline{v}'$
2. $(c\underline{u})' \quad = \quad c\underline{u}'$
3. $(\underline{u} \cdot \underline{v})' \quad = \quad \underline{u}' \cdot \underline{v} + \underline{u} \cdot \underline{v}'$
4. $(\underline{u} x \underline{v})' \quad = \quad \underline{u}' \wedge \underline{v} + \underline{u} \wedge \underline{v}'$

Tangent vector to the curve:

$\underline{r}'(t) \quad = x'(t)\underline{i} + y'(t)\underline{j} + z'(t)\underline{k}$

1.5 Geometry of Space Curves-Curvature

Let $\underline{r}(t)$ be a vector description of a curve. Then the distance s(*t)* along the curve from the point $\underline{r}(t_0)$ to the point $\underline{r}(t)$ is, as we have seen, simply

$$s(t) \quad = \int_{t_0}^{t} |\,\underline{r}'(u)du;$$

Assuming, $\underline{r}'(t) \neq 0$

Now then the vector $\underline{T} \ = \ \dfrac{d\underline{r}}{dt} / \left|\dfrac{d\underline{r}}{dt}\right| \ = \ \dfrac{\underline{r}'(t)}{ds/dt} \ = \ \dfrac{d\underline{r}}{ds}$

is tangent to *R* and has length one. It is called the ***unit tangent vector.***

Consider next the derivative $\frac{d}{ds}(\underline{T}.\underline{T}) = \underline{T}.\frac{d\underline{T}}{ds} + \frac{d\underline{T}}{ds}.\underline{T} = 2\underline{T}.\frac{d\underline{T}}{ds}$

But we know that $\underline{T}.\underline{T} = |T|^2 = 1$. Thus $\underline{T}.\frac{d\underline{T}}{ds} = 0$, which means that the vector $\frac{d\underline{T}}{ds}$ perpendicular, or normal, to the tangent vector T. The length of this vector is called the *curvature* $(\kappa = \frac{1}{\rho})$ and is usually denoted by the letter κ. Thus $\kappa\,\underline{n} = \frac{d\underline{T}}{ds}$ and $\kappa = \left|\frac{d\underline{T}}{ds}\right|$

The unit vector $\underline{n} = \frac{1}{\kappa}\frac{d\underline{T}}{ds}$

is called the ***principal unit normal vector***, and its direction is sometimes called the ***principal normal direction.***

Example:
Consider the circle of radius *a* and center at the origin: $\underline{r}(t) = a\cos t\,\underline{i} + a\sin t\,\underline{j}$

Example:

If $\underline{r}(t) = (t+1)\underline{i} + 2t\,\underline{j} + t^2\underline{k}$, find unit tangent vector to the curve.

Torsion of a Space Curve

Let R(*t)* be a vector description of a curve. If $\underline{T}$ is the unit tangent and $\underline{n}$ is the principal unit normal, the unit vector $\underline{b} = \underline{T} \times \underline{n}$ is called the ***binormal vector***. Note that the binormal is orthogonal to both $\underline{T}$ and $\underline{n}$. Let's see about its derivative $\frac{d\underline{b}}{ds}$ with respect to arc length s. First, note that $\underline{b}.\underline{b} = 1 = |\underline{b}|^2$, and so $\underline{b}.\frac{d\underline{b}}{ds} = 0$, which means that being orthogonal to , the derivative $\frac{d\underline{b}}{ds}$ is in the plane of $\underline{T}$ and $\underline{n}$. Next, note that $\underline{b}$ *is* perpendicular to the tangent vector $\underline{T}$, and so $\underline{b}.\underline{T} = 0$. Thus $\underline{T}.\frac{d\underline{b}}{ds} = 0$. So what have we here? The vector $\frac{d\underline{b}}{ds}$ is perpendicular to both $\underline{b}$ and $\underline{T}$, and so must have the direction of $\underline{n}$. This means

$\frac{d\underline{b}}{ds} = -\tau\underline{n}$ and $\left|\frac{d\underline{b}}{ds}\right| = |\tau|$. The scalar τ is called the ***torsion.***

Example:
Let $\underline{r}(t) = a\cos t\,\underline{i} + a\sin t\,\underline{j} + ct\,\underline{k}$ be a space curve, which is represented by a circular helix. Find unit tangent vector and curvature to the curve.

Exercises:

1. (a)Find a vector tangent to the curve $f(t)=(t^2+t)\underline{i}+(t+1)\underline{j}-(t_3+5)\underline{k}$ at the point (1, 1, 0).

   (b)Find a vector equation for the line tangent to this same curve at the point (1, 1, 0).

2. The position of a particle is given by $\underline{r}(t)=\cos(t^3)\underline{i}+\sin(t^3)\underline{j}$.

   (a)Find the velocity of the particle.

   (b)Find the speed of the particle.

3. Let L be the straight line passing through the point (5, 0, 3) in the direction of the vector $\underline{a}=\underline{i}+2\underline{j}-\underline{k}$, and let M be the straight line passing through the point (0, 0, 6) in the direction of $\underline{b}=\underline{i}-3\underline{j}+2\underline{k}$.

   a) Are L and M parallel? Explain.

   b) Do L and M intersect? Explain.

4. Let L be the straight line passing through the point (1, 1, 3) in the direction of the vector $\underline{a}=\underline{i}+2\underline{j}-\underline{k}$, and let M be the straight line passing through the point (0, 1, 5) in the direction of $\underline{b}=\underline{i}-3\underline{j}+2\underline{k}$ **.** Find the distance between L and M.

5. Find the length of the arc of the curve $\underline{r}(t)=3\cos t\underline{i}+3\sin t\underline{j}+4t\underline{k}$ between the points (3, 0, 0) and (3, 0, 16π).

6. Find the unit tangent vector T the principal normal $\underline{n}$, Curvature κ for the curves

(a) $\underline{r}(t) = 5\cos t\underline{i}+5\sin t\underline{j}+2t\underline{k}$

(b) $\underline{r}(t) = (2t+3)\underline{i}+(5-t^2)\underline{j}$

(c) $\underline{r}(t) = e^t\cos t\underline{i}+e^t\sin t\underline{j}+6\underline{k}$

6. Find the curvature of $\underline{r}(t) = t\underline{i}+t^3\underline{j}$, at what point on the curve is the curvature the largest? Smallest?

7. Find the binomial and torsion for the curve $\underline{r}(t) = 4\cos t\underline{i}+3\sin t\underline{j}$.

8. Find the binomial and torsion for the curve $\underline{r}(t) = \frac{\sin t}{\sqrt{2}}\underline{i}+\cos t\underline{j}+\frac{\sin t}{\sqrt{2}}\underline{k}$