

Lecture 1

CALCULUS

1. Introduction

1.1 What is vector calculus?

Vector calculus is how to define and measure the variation of temperature, fluid velocity, force, magnetic flux etc. over all three dimensions of space. In the real 3D engineering world, one wants to know things like the stress and strain inside a structure, the velocity of the air flow over a wing, or the induced electromagnetic field around an aerial. For such questions, it is simply not good enough to deal with $\frac{dy}{dx}$ and $\int f(x)dx$. We must instead know how to integrate and differentiate *vector* quantities with three components (in directions i, j and k) which depend on three co-ordinates x, y, z . Vector calculus provides the necessary mathematical notation and techniques for dealing with such issues. First, let's recall what we mean by vectors and calculus in isolation.

1.2 Vectors (revision)

- Notation: $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k} = \langle v_1, v_2, v_3 \rangle$
 - length: $|\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ unit vector: $\underline{v} = \frac{\underline{v}}{|\underline{v}|}$
- Position vector: $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = \langle x, y, z \rangle$

1.3 Scalar field, vector field and Scalar functions

1. A **scalar function** (of one variable) $f(x)$ or $f(t)$ is a formula that takes a scalar and returns a scalar. It might be used to describe the spatial variation of temperature $T(x)$ along a one-dimensional bar heated at one end, or the time variation of the DC current $i(t)$ across a certain component in an electrical circuit.

2. A **scalar field** ϕ is a scalar quantity defined over a region of space. It takes a vector (of positions) and returns a scalar.
 $\phi = f(x, y, z) = f(r)$ (or $f(x, y)$ in 2D).

Eg: The variation of temperature $T(x, y, z)$ in this room using Cartesian co-ordinates. We might also think of the variation of density or charge density $\rho(x, y, z)$ inside a solid object.

3. A **vector field** $\underline{v}(x, y, z)$ is a vector-valued quantity defined over a region of space. It is defined by a function that takes a vector (of positions) and returns a vector
 $\underline{v} = v_1(x, y, z)\underline{i} + v_2(x, y, z)\underline{j} + v_3(x, y, z)\underline{k}$

$$\underline{v} = v_1(x, y)\underline{i} + v_2(x, y)\underline{j} \quad (\text{in 2D})$$

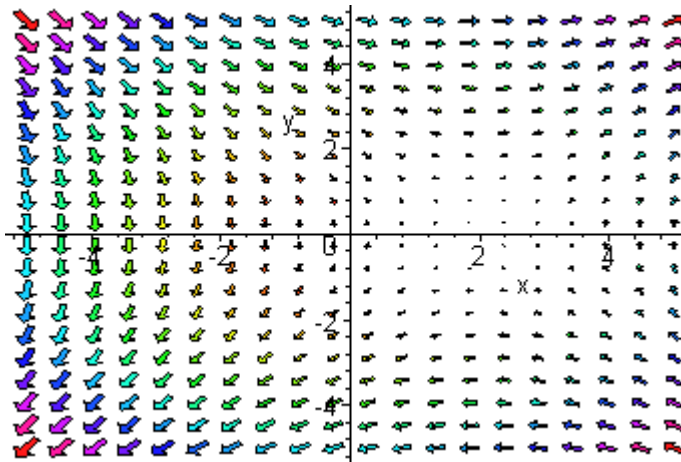
Eg: The spatial variation of fluid velocity $\underline{v}(x, y, z)$ in a steady flow, or current $I(x, y, z)$ flowing in a conductor.

Example:

An important vector field that we have already encountered is the *gradient vector field*. Let $f(x, y)$ be a differentiable function then the function that take a point (x_0, y_0) to $\text{grad}f(x_0, y_0)$ is a vector field since the gradient of a function at a point is a vector.

For example, if $f(x, y) = 0.1xy - 0.2y$ then $\text{grad}f(x, y) = 0.1y\underline{i} + (0.1x - 0.2)\underline{j}$

The sketch of the gradient is pictured below.



1.4 Vector functions

A **vector function** (of one variable) $\underline{v}(x)$ or $\underline{v}(t)$ takes a scalar and returns a vector:

$\underline{v} = v_1(t)\underline{i} + v_2(t)\underline{j} + v_3(t)\underline{k}$. Such functions might be used to describe the motion of a particle whose position vector \underline{r} is $\underline{r}(t)$ at time t ; or the external forces $\underline{F}(x)$ acting at distance x along a 1-dimensional case.

Differentiation and integration of vector functions are easy! One simply differentiates or integrates the components separately.

$$\frac{d}{dt}\underline{v} = \frac{d}{dt}v_1(t)\underline{i} + \frac{d}{dt}v_2(t)\underline{j} + \frac{d}{dt}v_3(t)\underline{k}$$

Example:

A particle moves on a circle of radius 1, such that its position vector is given by

$$\underline{r}(t) = \sin t\underline{i} + \cos t\underline{j}$$

Calculate its velocity and acceleration. Show that the velocity and acceleration are orthogonal.

Example:

Sketch the graph of the following vector function.

$$\vec{r}(t) = \{t, 1\}$$

- **Rules of differentiation**

. Here $\underline{u} = \underline{u}(t)$, $\underline{v} = \underline{v}(t)$ and c is a constant

$$1. (\underline{u} + \underline{v})' = \underline{u}' + \underline{v}'$$

$$2. (c\underline{u})' = c\underline{u}'$$

$$3. (\underline{u} \cdot \underline{v})' = \underline{u}' \cdot \underline{v} + \underline{u} \cdot \underline{v}'$$

$$4. (\underline{u} \times \underline{v})' = \underline{u}' \wedge \underline{v} + \underline{u} \wedge \underline{v}'$$

Tangent vector to the curve:

$$\underline{r}'(t) = x'(t)\underline{i} + y'(t)\underline{j} + z'(t)\underline{k}$$

1.5 Geometry of Space Curves-Curvature

Let $\underline{r}(t)$ be a vector description of a curve. Then the distance $s(t)$ along the curve from the point $\underline{r}(t_0)$ to the point $\underline{r}(t)$ is, as we have seen, simply

$$s(t) = \int_{t_0}^t |\underline{r}'(u)| du;$$

Assuming, $\underline{r}'(t) \neq 0$

$$\text{Now then the vector } \underline{T} = \frac{d\underline{r}}{dt} / \left| \frac{d\underline{r}}{dt} \right| = \frac{\underline{r}'(t)}{ds/dt} = \frac{d\underline{r}}{ds}$$

is tangent to R and has length one. It is called the **unit tangent vector**.

$$\text{Consider next the derivative } \frac{d}{ds}(\underline{T} \cdot \underline{T}) = \underline{T} \cdot \frac{d\underline{T}}{ds} + \frac{d\underline{T}}{ds} \cdot \underline{T} = 2\underline{T} \cdot \frac{d\underline{T}}{ds}$$

But we know that $\underline{T} \cdot \underline{T} = |\underline{T}|^2 = 1$. Thus $\underline{T} \cdot \frac{d\underline{T}}{ds} = 0$, which means

that the vector $\frac{d\underline{T}}{ds}$ perpendicular, or normal, to the tangent vector

\underline{T} . The unit vector along $\frac{d\underline{T}}{ds}$ denotes by \underline{n} . $\underline{n} = \frac{\frac{d\underline{T}}{ds}}{\left| \frac{d\underline{T}}{ds} \right|}$. The

length of this vector is called the *curvature* ($\kappa = \frac{1}{\rho}$) and is usually

denoted by the letter \mathcal{K} . Thus $\kappa \underline{n} = \frac{d\underline{T}}{ds}$ and $\kappa = \left| \frac{d\underline{T}}{ds} \right|$. The unit

vector $\underline{n} = \frac{1}{\kappa} \frac{dT}{ds}$ is called the *principal unit normal vector*, and its direction is sometimes called the *principal normal direction*.

Example:

Consider the circle of radius a and center at the origin:

$$\underline{r}(t) = a \cos t \underline{i} + a \sin t \underline{j}$$

Example:

If $\underline{r}(t) = (t+1)\underline{i} + 2t\underline{j} + t^2\underline{k}$, find unit tangent vector to the curve.

Torsion of a Space Curve

Let $\underline{R}(t)$ be a vector description of a curve. If \underline{T} is the unit tangent and \underline{n} is the principal unit normal, the unit vector $\underline{b} = \underline{T} \times \underline{n}$ is called the *binormal vector*. Note that the binormal is orthogonal to both \underline{T} and \underline{n} . Let's see about its derivative $\frac{d\underline{b}}{ds}$ with respect to arc length s .

First, note that $\underline{b} \cdot \underline{b} = 1 = |\underline{b}|^2$, and so $\underline{b} \cdot \frac{d\underline{b}}{ds} = 0$, which means that being orthogonal to \underline{b} , the derivative $\frac{d\underline{b}}{ds}$ is in the plane of \underline{T} and \underline{n} .

Next, note that \underline{b} is perpendicular to the tangent vector \underline{T} , and so $\underline{b} \cdot \underline{T} = 0$. Thus $\underline{T} \cdot \frac{d\underline{b}}{ds} = 0$. So what have we here? The vector $\frac{d\underline{b}}{ds}$ is perpendicular to both \underline{b} and \underline{T} , and so must have the direction of \underline{n} . This means

$$\frac{d\underline{b}}{ds} = -\tau \underline{n} \quad \text{and} \quad \left| \frac{d\underline{b}}{ds} \right| = |\tau|. \quad \text{The scalar } \tau \text{ is called the}$$

torsion.

Example:

Let $\underline{r}(t) = a \cos t \underline{i} + a \sin t \underline{j} + ct \underline{k}$ be a space curve, which is represented by a circular helix. Find unit tangent vector, torsion and curvature to the curve.