Lecture 2

2.1 The gradient of a scalar field

Consider the level curves of a scalar field in 2D.
Clearly the gradient or slope is different depending on the path taken over the contours. We can calculate the gradient or rate of change of slope along any path. So clearly the differential (derivative) of a ***scalar*** field must itself be a ***vector*** (it has magnitude and direction).

Definition: The gradient of scalar field $\phi(x,y,z)$ usually denote by

$$grad(\phi) = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k} \quad (Vector)$$

What is grad ϕ?

(i) Its magnitude gives the slope (rate of change) of ϕ, when moving along a certain direction. This direction is the **normal vector** to the surface

$$\phi = const.$$

Proof:

In other words grad ϕ is orthogonal to the tangent to any curve lying in the surface $\{\phi = c\}$. Therefore it defines the normal vector to the level surface.

(ii) Its magnitude gives the rate of change of ϕ. To see this we need to define the concept of a directional derivative

Definition: The **directional derivative** $\mathbf{D_a}$ of a scalar field ϕ (x,y,z) at a point P is the differential of $\phi(x,y,z)$ at P in the direction of the unit vector $\underline{a}$.

$$D_{\underline{a}}\phi = \frac{d\phi}{dt} = \underline{a}.grad(\phi)$$

Now, since $\underline{a}.grad(\phi) = |\underline{a}||grad(\phi)|\cos\theta$

We have that $D_{\underline{a}}\phi$ largest when $\theta = 0$, and in this case $|D_{\underline{a}}\phi| = |grad(\phi)|$.

<u>Thus,</u>

$|grad(\phi)|$ Points in direction $\theta = 0$ in which ϕ increases the most.

In fact, $grad(\phi(x,y,z))$ defines a vector field. To stress that grad ϕ is a vector field, we have a special notation

$$grad(\phi) = \nabla\phi \quad \text{Where } \nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$$

∇ ***'del' is the vector differential operator***

Example:

If $\underline{A}=x^2z\underline{i}+xy\underline{j}+y^2z\underline{k}$ and $\underline{B}= yz^2\underline{i}+xz\underline{j}+x^2z\underline{k}$ .Find $\nabla(\underline{A}.\underline{B})$.

2.2 Applications of gradient

1. Equation for the tangent plane to a surface

∇f is perpendicular to level surfaces of functions $f(x, y, z)$. *So* therefore if we can write a surface as $f(x, y, z) = c$, then the unit normal is

$$\underline{\hat{n}} = \frac{\nabla f}{|\nabla f|},$$

So the equation for the tangent plane at a point P with position vector $\underline{r} = \underline{r}_o$ is

$$(\underline{r}-\underline{r}_0).\underline{\hat{n}} = 0 \quad \Rightarrow \quad (\underline{r}-\underline{r}_0).\nabla f\big|_{r=r_0} = 0$$

Example: Show that, the equation for the tangent plane to a sphere of radius a at a point (x_0, y_o, z_o) is $xx_o + yy_o + zz_o = a^2$

2. Stationary points of a Surface

$\nabla f = 0$ defines the points at which the function f (x, y, z) has its stationary points = (x_o, y_o, z_o) such that

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Bigg|_{x=x_o y=y_o, z=z_o} = 0 \Rightarrow \quad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

(Conditions for stationary points)

In 2D, we know there are three kinds of stationary points. Maxima, minima and saddles.

3. Force and potential energy

We know for 1D that `$\mathbf{F} = \frac{dv}{dx}$' where V is potential energy and **F** applied force. How does this apply in more dimensions?

F = grad V

i.e. force is in the direction of maximum increase in potential.

Example: A spaceship moves in the gravitational field of a planet with gravitational potential $\phi = \frac{k}{|\underline{r}|}$ k is a constant.

Calculate the magnitude and the direction of the force grads acting on the ship at a distance r from the centre of the planet.

Divergence and Curl of a Vector Field

There are two key notions of differentiation of a vector field $\underline{v}(x, y, z)$. They are the so-called divergence div$\underline{v}$ and curl curlv (sometimes called rot$\underline{v}$).

Curl represents the amount to which "a particle being carried by the vector field is being rotated"

Divergence: $div(\underline{v}) = \nabla.\underline{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).(v_1, v_2, v_3)$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Curl: $curl(\underline{v}) = \nabla x \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\underline{i} - \left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right)\underline{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\underline{k}$

Note that the direction of curl v is the axis about which the vector field is being rotated (in an anticlockwise sense).

Example: Calculate div(**v**) and curl(**v**) for the vector fields
1. **v** = 4xy$\underline{i}$ + yz$\underline{j}$ + x$\underline{k}$
2. $\underline{r}$ = xi + y j + zk (=position vector $\underline{r}$)

Definition: A vector field is said to be (Solenoidal) **Incompressible** if **div v=0** at all points. i.e. for all values of x,y,z is called solenoid vector field. Conversely, if **A** is **a** solenoidal vector field then there exist vector **G** so that **A** = curl **G** is vector potential of **A**

Example: This has a natural interpretation in fluid mechanics, where the equation of continuity states that the fluid density $\rho(r)$ (a scalar field) and the fluid velocity $\underline{v}(r)$ are linked by the continuity equation

$$\frac{\partial \rho}{\partial t} + div\rho\underline{v} = 0$$

So that if the fluid has constant density (e.g. water, to good approximation, but not air) we have $\rho = \rho_o$=const., and hence

$$div\rho_o\underline{v} = \rho_0 div\underline{v} = 0 \Rightarrow div\underline{v} = 0$$

Incompressible vector fields are also called **solenoidal**

Irrotational flow or conservative forces

Definition:
A fluid flow that whose velocity field **v** is curl free, and curl v = 0 is called **irrotational.**

Definition:
A force field F that satisfies curl$\underline{F}$=0 is said to be conservative.
More generally it can be shown that curl v = 0 if and only if v = gradϕ, for some scalar field ϕ.
ϕ is called the **scalar potential** of a **conservative vector field.**

Example
Show that the following vector fields are conservative, and find their scalar fields so that $\underline{F} = \nabla\phi$
(i) $\underline{F} = (2xy + z^2)\underline{i} + x^2\underline{j} + 3xz^2\underline{k}$
(ii) $\underline{F} = 2x\underline{i} + 4y\underline{j} + 8z\underline{k}$