Vector Operators: Grad, Div and Curl

We introduce three field operators which reveal interesting collective field properties.

- the **gradient** of a scalar field,
- the **divergence** of a vector field, and
- the **curl** of a vector field.

There are two points to get over about each:

- The mechanics of taking the grad, div or curl, for which you will need to brush up your multivariate calculus.
- The underlying physical meaning.

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The gradient of a scalar field

If $U(x, y, z)$ is a scalar field, ie a scalar function of position $\underline{r} = \langle x, y, z \rangle$ in 3 dimensions, then its **gradient** at any point is defined in Cartesian co-ordinates by $gradU = \frac{\partial U}{\partial x}\underline{i} + \frac{\partial U}{\partial y}\underline{j} + \frac{\partial U}{\partial z}\underline{k}$.

It is usual to define the **vector operator** which is called "del" or "nabla".

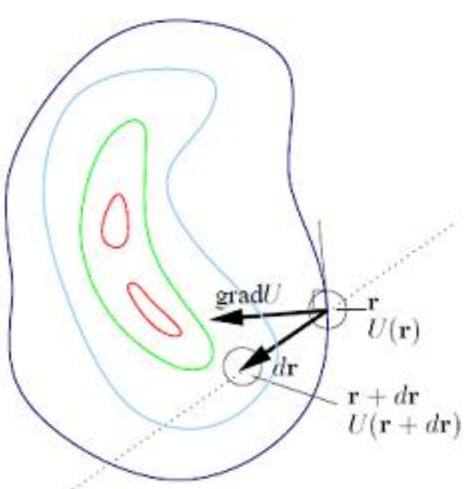
$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k} .$$

Then $gradU \equiv \nabla U$.

Note: ∇U is a vector field.

We can see that the gradient of a scalar field tends to point in the direction of greatest change of the field. .

The significance of grad



Here $\underline{r}$ in some scalar field U and we move an infinitesimal distance $d\underline{r}$ we know that the change in U is $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$

But we know that $d\underline{r} = \langle dx, dy, dz \rangle$ and $\nabla U = \frac{\partial U}{\partial x}\underline{i} + \frac{\partial U}{\partial y}\underline{j} + \frac{\partial U}{\partial z}\underline{k}$ so that the change in U is also given by the scalar product $dU = \nabla U . d\underline{r}$. Now divide both sides by ds

$$\frac{dU}{ds} = \nabla U . \frac{d\underline{r}}{ds}$$

But remember that $|d\underline{r}| = ds$ so $d\underline{r}/ds$ is a unit vector in the direction of $d\underline{r}$.

So $gradU$ has the property that the rate of change of U wrt distance in a particular direction ($\hat{d}$) is the projection of $gradU$ onto that direction (or the component of $gradU$ in that direction).
In other words grad U is orthogonal to the tangent to any curve lying in the surface $\{U = c\}$. Therefore it defines the normal vector to the level surface.

Proof:

The quantity dU/ds is called a **directional derivative**, but note that in general it has a different value for each direction, and so has no meaning until you specify the direction.

Applications of gradient

1. Equation for the tangent plane to a surface

∇f is perpendicular to level surfaces of functions $f(x, y, z)$. *So* therefore if we can write a surface as $f(x, y, z) = c$, then the unit normal is

$$\underline{\hat{n}} = \frac{\nabla f}{|\nabla f|},$$

So the equation for the tangent plane at a point P with position vector $\underline{r} = \underline{r}_o$ is

$$(\underline{r} - \underline{r}_0).\underline{\hat{n}} = 0 \quad \Rightarrow \quad (\underline{r} - \underline{r}_0).\nabla f \Big|_{r=r_0} = 0$$

Example: Show that, the equation for the tangent plane to a sphere of radius a at a point (x_0, y_o, z_o) is $xx_o + yy_o + zz_o = a^2$

2. Stationary points of a Surface

$\nabla f = 0$ defines the points at which the function f (x, y, z) has its stationary points = $(x_o,$

y_o, z_o) such that

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)\Bigg|_{x=x_o y=y_o, z=z_o} = 0 \Rightarrow \qquad \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

(Conditions for stationary points)

In 2D, we know there are three kinds of stationary points. Maxima, minima and saddles.

3. Force and potential energy

We know for 1D that `**F** = $\frac{d\underline{v}}{dx}$' where V is potential energy and **F** applied force. How does this apply in more dimensions?

F = grad V

i.e. force is in the direction of maximum increase in potential.

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Example: A spaceship moves in the gravitational field of a planet with gravitational potential $\phi = \frac{k}{|\underline{r}|}$ k is a constant.

Calculate the magnitude of the force acting on the ship at a distance r from the centre of the planet.

The divergence of a vector field

The divergence computes a scalar quantity from a vector field by differentiation.

Divergence: $div(\underline{v}) = \nabla . \underline{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) . (v_1, v_2, v_3)$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Notice that the divergence of a vector field is a scalar field

The significance of div

Consider a vector field, water flow, and denote it by $v(\underline{r})$. This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\underline{v}$ per unit time.

Now take an infinitesimal volume element dV and figure out the balance of the flow of $\underline{v}$ in and out of dV.

To be specific, consider the volume element $dV = dxdydz$ in Cartesian co-ordinates.

So we see that,

The divergence of a vector field represents the flux generation per unit volume at each point of the field.

The curl of a vector field

$$curl(\underline{v}) = \nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Some definitions involving div, curl and grad

- A vector field with zero divergence is said to be **solenoidal**.
- A vector field with zero curl is said to be **irrotational**
- A scalar field with zero gradient is said to be **constant**.

Definition: A vector field is said to be (Solenoidal) **Incompressible** if **div v=0** at all points. i.e. for all values of x,y,z is called solenoidal vector field. Conversely, if **A** is **a** solenoidal vector field then there exist vector **G** so that **A** = culr**G. G** is vector potential of **A**

Example: This has a natural interpretation in fluid mechanics, where the equation of continuity states that the fluid density $\rho(r)$ (a scalar field) and the fluid velocity $\underline{v}(r)$ are linked by the continuity equation

$$\frac{\partial\rho}{\partial t} + div\rho\underline{v} = 0$$

So that if the fluid has constant density (e.g. water, to good approximation, but not air) we have $\rho = \rho_o$=const., and hence $div\rho_o\underline{v} = \rho_0 div\underline{v} = 0 \Rightarrow div\underline{v} = 0$

Incompressible vector fields are also called **solenoidal**

Definition:

A force field F that satisfies $curlF = 0$ is said to be conservative.

More generally it can be shown that $curlF = 0$ if and only if $F = grad\phi$, for some scalar field ϕ.

ϕ is called the **scalar potential** of a **conservative vector field.**

Proof:

First, we show that v = grad$\phi \Rightarrow curl\underline{v} = 0$

This is by direct calculation: $\underline{v} = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$

$$So,\ curl\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial x} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial z\partial y} - \frac{\partial^2\phi}{\partial y\partial z}\right)\underline{i} + \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right)\underline{j} + \left(\frac{\partial^2\phi}{\partial y\partial x} - \frac{\partial^2\phi}{\partial x\partial y}\right)\underline{k}$$

$$= 0\underline{i} + 0\underline{j} + 0\underline{k} = 0$$

Next, we show that $curl\underline{v} = 0 \Rightarrow \underline{v} = \nabla\phi$

for some scalar field $\phi(r)$

$$\text{Now curl } \underline{v} = 0 \Rightarrow \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} = 0 \quad (2.3)$$

$$\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} = 0 \quad (2.4)$$

$$\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad (2.5)$$

Now, taking $\frac{\partial}{\partial x}$ of (2.3) and $\frac{\partial}{\partial y}$ of (2.4) we get two expressions for $\frac{\partial^2 v_3}{\partial x\partial y}$. Setting these equal gives

$$\frac{\partial^2 v_2}{\partial z\partial x} = \frac{\partial^2 v_1}{\partial z\partial y} := \frac{\partial^3\phi}{\partial x\partial y\partial z} \quad (2.6)$$

For some scalar function $\phi(x, y, z)$. This can only be true if

$$\underline{v}_1 = \frac{\partial\phi}{\partial x} + b_1^/ y + c_1^/ z + d_1^/ \quad \text{and} \quad \underline{v}_2 = \frac{\partial\phi}{\partial y} + b_2^/ y + c_2^/ z + d_2^/$$

For some functions $b_1^/(x),\ c_1^/(x),\ d_1^/(x)$ and $b_2^/(y),\ c_2^/(y),\ d_2^/(y)$. Where $^/$ stands for "differentiate" But, by redefining ϕ to be

$$\phi \to \phi - b_1 y - c_1 z - d_1 x - b_2 xy - c_2 yz - d_2 y$$

We can without loss of generality choose $b_{1,2} = c_{1,2} = d_{1,2} = 0$ and

$$\frac{\partial^2 v_3}{\partial x\partial y} = \frac{\partial^2 v_2}{\partial x\partial z} := \frac{\partial^3\phi}{\partial x\partial y\partial z}$$

So that, without loss of generality $v_2 = \frac{\partial\phi}{\partial y},\ v_3 = \frac{\partial\phi}{\partial z}$

$$\Rightarrow (v_1, v_2, v_3) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \nabla\phi.$$

Example

Show that the following vector fields are conservative, and find their scalar fields so that $\underline{F} = \nabla\phi$

(i) $\underline{F} = (2xy + z^3)\underline{i} + x^2\underline{j} + 3xz^2\underline{k}$

(ii) $\underline{F} = 2x\underline{i} + 4y\underline{j} + 8z\underline{k}$