LECTURE 3
3.1Rules of Vector Differentiation

We have defined three kinds of derivatives involving the operator ∇

$$grad(\phi) = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}\,, \qquad div(\underline{v}) = \nabla.\underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\,,$$

$$curl(\underline{v}) = \nabla\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The good news is that you can apply all the usual formualae for differentiation with $\frac{d}{dx}$ replced by ∇ provided you are careful. This is because grad and curl are vectors, wherer as div is a scalar. Also div and curl apply to vector fields, where as grad applies to scalar fields.

In this lecture we look at more complicated identities involving vector operators. The main thing to appreciate it that the operators behave both as vectors and as differential operators, so that the usual rules of taking the derivative of, say, a product must be observed.

Let $\underline{u}(r)$ and $\underline{v}(r)$ be vector fields, $f(r)$ and $g(r)$ be scalar fields and α and β be constants.

1 Differentiation is Linear

$$grad(\alpha f + \beta g) = \alpha\nabla f + \beta\nabla g = \alpha grad(f) + \beta grad(g)$$

$$div(\alpha\underline{u} + \beta\underline{v}) = \alpha div(\underline{u}) + \beta div(\underline{v})$$

$$curl(\alpha\underline{u} + \beta\underline{v}) = \alpha curl(\underline{u}) + \beta curl(\underline{v})$$

$$grad(fg) = fgrad(g) + ggrad(f)$$

$$div(f\underline{u}) = fdiv(\underline{u}) + grad(f).(\underline{u})$$

$$curl(f\underline{u}) = f\nabla x\underline{u} + (\nabla f)x\underline{u} = fcurl(\underline{u}) + grad(f)x(\underline{u})$$

2 Vector Product Rule

$$div(\underline{u}x\underline{v}) = \underline{v}.curl(\underline{u}) - \underline{u}.curl(\underline{v})$$

$$curl(\underline{u}\times\underline{v}) = u(div(\underline{v})) - \underline{v}(div(\underline{u})) + (\underline{v}.\nabla)\underline{u} - (\underline{u}.\nabla)\underline{v}$$

$$grad(\underline{u}.\underline{v}) = (\underline{u}.\nabla)\underline{v} + (\underline{v}.\nabla)\underline{u} + \underline{u}\times curl(\underline{v}) + \underline{v}\times curl(\underline{u})$$

3. Vector Multiple Operations

$$div(grad(f)) = \nabla.\nabla(f) = \nabla^2 f$$

$$curl(grad(f)) = \nabla x\nabla(f) = 0$$

$div(curl(\underline{u})) = \nabla.\nabla x(\underline{u}) = 0$

$curl(curl(\underline{u})) = \nabla(\nabla.(\underline{u}) - \nabla^2 \underline{u}$

Example: James Clerk Maxwell established a set of four vector equations which are fundamental to working out how electromagnetic waves propagate. The entire telecommunications industry is built on these.

$divD = \rho$

$div\mathbf{B} = 0$

$curlE = -\frac{\partial}{\partial t} B$

$curlH = J + \frac{\partial}{\partial t} D$

In addition, we can assume the following, $B = \mu_r \mu_0 H,\ J = \sigma E,\ D = \varepsilon_r \varepsilon_0 E$, where all the scalars are constants. Now show that in a material with zero free charge density, $\rho = 0$, and with zero conductivity, $\sigma = 0$, the electric field E must be a solution of the wave equation $\nabla^2 E = \mu_r \mu_0 \varepsilon_r \varepsilon_0 (\partial^2 E / \partial t^2)$

Example: Let $\underline{v} = 3xyz^2 \underline{i} + 2xy^3 \underline{j} - x^2 yz \underline{k}, \quad \phi = 3x^2 - yz$

Find (i) $div(\underline{v})$ (ii) $curl(\underline{v})$ (iii) $\underline{v}.grad(\phi)$ and hence (iv) $div(\phi \underline{v})$ (v) $curl(\phi \underline{v})$.

3.2 Line Integrals

We are used to calculating integrals with respect to a set of coordinate axes, usually Cartesian. For example Area under a curve $f(x)$ between a and b = $\int_a^b f(x)dx$

In this section we will generalize this idea to *line integrals* which are entirely analogous to the above idea. Here, we allow the ‘axis’ of the integral to be an arbitrary curve in space and the function *f* will vary along that line. As an example of this, suppose that the temperature of our body at a particular time is $T(x,y, z)$. Now suppose we wish to calculate the average temperature of the blood in a particular vein. This vein traces out a curve through our body and we wish to average the temperature along that curve. This involves a line integral like the ones we will consider here. We will consider two types of line integrals: integrals of scalar functions and integrals of vector functions. As we will see, once we have parameterized the integral using *t* on [*a*,*b*], the integral in both cases reduces to an ordinary integral over [*a*,*b*]. This can be evaluated using integration rules.

Line integrals of scalar functions

Suppose we wish to integrate $f(x,y, z)$ along a curve *C*. Letting *s* be the arclength along the curve, we wish to find

$\int_C f(x,y,z)ds$.

The arc length *s* has units of length and the integral above is the “area” under the curve C along that curve. However, the integral as it is written above cannot be evaluated since we do not have an expression for *x*, *y* and *z* along C. To get this, suppose that the curve C is parameterized using the parameter *t*, then we can write the integral above as $\int_{t_0}^{t_1} f(x(t),y(t),z(t))\frac{ds}{dt}dt$. Where we have used the Chain Rule, and *t* ranges from t_0 to t_1.

Where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the parametric form of the curve C.

Examples of line integrals of scalar functions

An important special case of the line integral above is where $f(x,y,z) = 1$, which defines the arc length of the curve C. Another important application is when $f(x,y,z)$ represents the linear density (mass per unit distance), for example along a wire. Then the line integral defines the total mass of the wire.

- To ensure the integral is defined, we assume that the curve C is smooth, or piecewise smooth, ie continuous with finitely many smooth pieces.
- The value of the scalar line integral is independent of the parameterization used to define C. In particular, the integral has the same value if the direction (orientation) of the parameterization is reversed.
- If C is a closed path, sometimes $\oint_C$ is written instead of $\int_C$.
- Following results hold for line integral.
    1. $\int_C kf ds = k\int_C f ds$
    2. $\int_C (f+g)ds = \int_C f ds + \int_C g ds$
    3. $\int_{C_1+c_2} (f+g)ds = \int_{C_1} f ds + \int_{C_2} g ds$

The line integral of a vector function

Suppose a particle is moved along a curve C from $\mathbf{r}(a)$ to $\mathbf{r}(b)$ by a force $\mathbf{F}(\mathbf{r})$. What is the work done W by the force $\mathbf{F}$ on the particle? $W = \int_a^b \underline{F}.d\underline{r}$

This expression holds for any vector function, not just force as in the derivation above. Some notes about this line integral are listed below.

- To ensure the integral is defined, we assume that C is smooth or piecewise smooth.
- The value of the line integral of a vector function is independent of the parameterization used to define the curve as long as the direction (orientation) is the same. If the direction is reversed, then the value of the integral is multiplied by -1.
- If C is a closed path then $\int_C$ is often written $\oint_C$
- The line integral has the following properties:

a. $\int_C kFds = k\int_C Fds$

b. $\int_C (F+G)ds = \int_C Fds + \int_C Gds$

c. $\int_{C_1+c_2} (F+G)ds = \int_{C_1} Fds + \int_{C_2} Gds$

Parametrisation of Curves

The key to evaluating such integrals is to define a single co-ordinate t that parametrises the curve C. Consider the curves in 2D. For some curves it is obvious how to do this, eg. Use the x-coordinate as the parameter:

- Straight line $y = a + bx \Rightarrow x = t,\ y = a + bt$ or $\underline{r}(t) = (t, a + bt)$
- Parabola $y = a + bx^2 \Rightarrow x = t,\ y = a + bt^2$ or $\underline{r}(t) = (t, a + bt^2)$

For other curves one can use an angular formulation

- **Circle** $x^2 + y^2 = a^2 \Rightarrow x = a\cos t,\quad y = a\sin t$ or $\underline{r}(t) = (a\cos t, a\sin t)$ and $d\underline{r}(t) = (-a\sin t,\ a\cos t)$
- **Ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow x = a\cos t,\quad y = b\sin t$ or $\underline{r}(t) = (a\cos t, b\sin t)$ and $d\underline{r}(t) = (-a\sin t,\ b\cos t)$
- Straight line $\underline{r} = \underline{a} + \underline{b}t$ or $\frac{x-a}{l} = \frac{x-b}{m} = \frac{x-c}{n} = t(say)$

Therefore $x = a + lt,\ y = b + mt,\ z = c + nt$

- Helix (shape of light spring) $\underline{r} = (a\cos t, a\sin t, ct)$

Here a is the radius of the helix and $\frac{b}{a} = \tan\theta$, where θ is the helix angle.

Evaluation of work Integrals

1 parametrise the curve C as $\underline{r}(t) = (x(t), y(t), z(t))$
2 work the limits a and b on t .
3 evaluate the vector field $\underline{v}$ along the (x,y,z)=(x(t),y(t),z(t)), form the dot product and integrate w.r.t t:

$$\int_C v(r).d\underline{r} = \int_{t=a}^{t=b} (v(x(t), y(t), z(t))).\left(\frac{dx(t)}{dt}\underline{i} + \frac{dy(t)}{dt}\underline{j} + \frac{dz(t)}{dt}\underline{k}\right)dt$$

Example1: Find the work done moving a particle from (0,0,0) to (1,1,1) in the field $F = (2x + y^2)\underline{i} - 3xy\underline{j} + \underline{k}$ along the straight path C_1, the straight line joining (0,0,0) to (1,1,1) and parametrically given by x= t, y =t^2, z = t^3.

Example2: Find the work done moving a particle from (0,0,0) to (1,1,1) in the field $F = (2xy + z^3)\underline{i} + x^2\underline{j} + 3xz^2\underline{k}$ along the following paths

a) the straight path C_1, the straight line joining (0,0,0) to (1,1,1) .

b) the path C_2 composed of the three straight lines joining (0,0,0) to (1,0,0) to (1,1,0) to (1,1,1).

Note that both answers are same as they would be for any curve C joining (0,0,0) and (1,1,1). This is because the above force field $\underline{F}$ is conservative.

3.5 Conservative Vector fields and independence of path

Recall from lecture 2 , a vector field is $\underline{v}$ conservative in a region , if its circulation along every closed curve in the region is zero. i.e. $\Rightarrow curl(\underline{v}) = 0 \Rightarrow \underline{v} = \nabla\phi$

The Fundamental Theorem of Line Integrals

Let **F** be a conservative vector field with potential function f, and C be any smooth curve starting at the point A and ending at the point B. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

To prove the fundamental theorem of line integrals we will use the following outcome of the chain rule:

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a vector valued function, then

$\frac{d}{dt} f(\mathbf{r}(t)) = f_x x'(t) + f_y y'(t)$We are now ready to prove the theorem. We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x,y) \cdot \mathbf{r}'(t)\, dt$$

$$= \int_a^b (f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j})\, dt$$

$$= \int_a^b (f_x(x,y)x'(t) + f_y(x,y)y'(t))\, dt$$

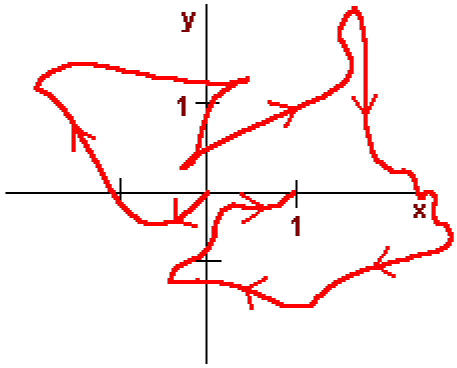
$$= \int_a^b \frac{d}{dt}\left(f(x(t), y(t))\right) dt = f(x(b), y(b)) - f(x(a), y(a))$$

$$= f(B) - f(A)$$

Theorem:
The necessary and sufficient condition for a continuous vector field $\underline{F}$ to be conservative (or irrotational) in a simply connected region R is $\underline{F} = \nabla\phi$ that it is the gradient of a scaler field.

Example: Find the work done by the vector field

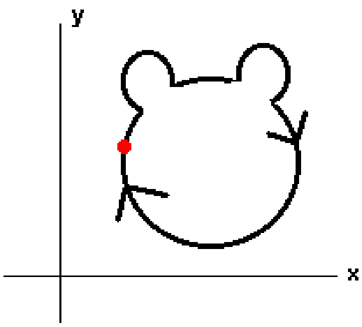
$\mathbf{F}(x,y) = (2x - 3y)\mathbf{i} + (3y^2 - 3x)\mathbf{j}$ along the curve indicated in the graph below



Independence of Path and Closed Curves

Example: Find the work done by the vector field

$\mathbf{F}(x,y) = (\cos x + y)\mathbf{i} + (x + e^{\sin y})\mathbf{j} + (\sin(\cos z))\mathbf{k}$ along the closed curve shown below



Theorem: Conservative Vector Fields and Closed Curves

Let **F** be a vector field with components that have continuous first order partial derivatives and let C be a piecewise smooth curve. Then the following three statements are equivalent

1. **F** is conservative.

2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

3. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed curves C.

Example:

Calculate $curl(\underline{F})$ for the force field $\underline{F}$ taken in worked in previous example, hence show that the field is conservative. Find the scalar field ϕ and hence calculate the work done in moving from (1,-2,1) to (3,1,4).

Remarks

- The course is also true; if the work integral is independent of path taken between any two points then there must exist a scalar potential function ϕ such that $\underline{F} = \nabla\phi$.