LECTURE 4

Double integrals

Definition:

Suppose D is a region in $R2$, $f(x, y)$ is a scalor function of two variable, we will define the double integral of f over D in the following way $\iint_R f(x,y)dxdy$ Double Integral is always used to calculate the volume of some solid in $R3$. The way to evaluate this double integral TOTALLY depends on the region D, basically we have the following 3 big cases.

1. D= [a,b]× [c,d] is rectangle, then $\iint_D f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$
2. D belongs to type I: $D: a \le x \le b \quad f_1(x) \le y \le f_2(x)$ or

   type II : $D: g_1(y) \le x \le g_2(y)$, $c \le y \le d$

3. D is easier to describe in polar coordinate system, for example when you have polar rectangles, like circles,a slice of the circle or even part of the slice, we should use polar integral, here the idea of change of variable is involved. (When you do change of variable, you need to change EVERYTHING, including the integral bounds, integrand, and the variable.)

Example: Compute each of the following double integrals over the indicated rectangles.

(a)

$$\iint_R \frac{1}{(2x+3y)^2}\,dA, \quad R = [0,1]\times[1,2]$$

(b)

$$\iint_R x\mathbf{e}^{xy}\,dA, \quad R = [-1,2]\times[0,1]$$

Example 1 Evaluate each of the following integrals over the given region D.

(a)

$$\iint_D \mathbf{e}^{\frac{x}{y}}\,dA,\ D = \{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\}$$

(b) $\iint_D 4xy - y^3\,dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

(c) $\iint_D 6x^2 - 40y\,dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Example 2: Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy\, dx$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$

Properties of double integrals

The following are more or less obvious from thinking of the integral as the volume under the height function f (x, y). If limits are constant (the region is rectangle) and f(x,y)= h(x)g(y) then

(i) $\int_a^b \int_c^d h(x)g(y)dxdy = \int_a^b h(x)dx \int_c^d g(y)dy$

(ii) $\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$

Application of Double Integrals

Given a distribution of mass in a region R of the (x,y) plane:

- **Area** $A = \iint_R dxdy$
- **Mass** $M = \iint_R \rho(x,y)dxdy$ where $\rho(x,y)$ is the density.
- The **Centre of Gravity** of the mass in R has co-ordinates $\bar{x}, \bar{y}$ where $\bar{x} = \frac{1}{M}\iint_R x\rho(x,y)dxdy$,

  $\bar{y} = \frac{1}{M}\iint_R y\rho(x,y)dxdy$

- The **Moment of Inertia** of the mass R about the x and y axes respectively.

  $I_x = \iint_R y^2\rho(x,y)dxdy \qquad I_y = \iint_R x^2\rho(x,y)dxdy$.

Double Integrals in Polar Coordinates

There are some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of a disk or ring. In these cases using Cartesian coordinates could be somewhat cumbersome.

That is $\iint_R f(x,y)dxdy = \iint_R f(r\cos\theta, r\sin\theta)rdrd\theta$

Example :

(a) Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$
(b) Evaluate the following integral by first converting to polar coordinates.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos\left(x^2 + y^2\right) dx\, dy$$

Transformation of Double integrals

More generally, if we change co-ordinates to some other co-ordinate system

x = f(u,v), y = y(u,v) mapping points (x,y) of the xy plane into points (u,v) of the uv plane. Region R mapped into region R'. Then we have

$$\iint_R F(x,y)dxdy = \iint_{R'} G(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$, $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the determinant of the so called **Jacobian Matrix**.

Taking the case of polar co-ordinates

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dr d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr d\theta \quad = r dr d\theta$$

Example:
Find the polar moment of inertia of the region in the xy plane bounded by $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $xy = 2$ and $xy = 4$ assuming unit density.

Ex:

(a) Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.

(b) $\iint_D 2xy\, dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.