Lecture4

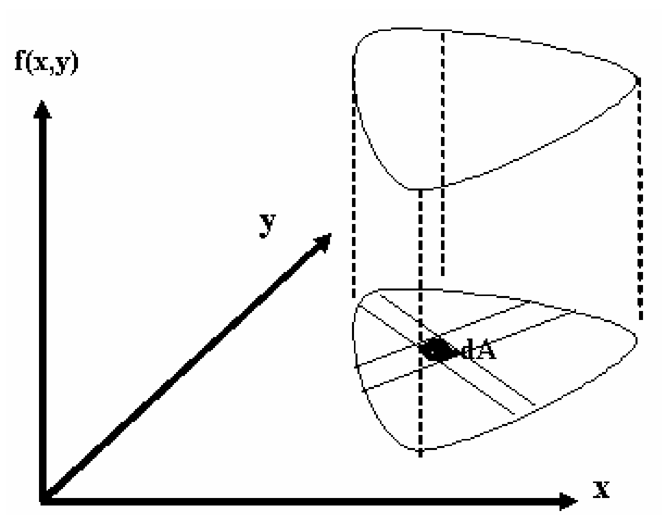
Double integrals

$\int_a^b f(x)dx$ = area below graph y = f(x) over [a,b].

Now: double integral $\iint_R f(x,y)dA =$ volume below graph z = f(x,y) over plane region R.

E.g. calculate the mass of a lamina occupying a region R in the (x, y)-plane, whose density variation is $\rho(x,y)$. Therefore, total mass $\iint_R \rho(x,y)dxdy.$

This is an example of a double integral, which we write in Cartesian co-ordinates as $\iint_R f(x,y)dxdy$ and interpret as representing the volume under the `height functionô $f(x,y)$ above the region R.



Such integrals are calculated by first integrating with respect to one variable, then the other.

NOTATION: we perform the inner integral first

$$\iint_R f(x,y)dxdy = \int \left[\int f(x,y)dx\right]dy. \qquad (4.1)$$

Example: Compute each of the following double integrals over the indicated rectangles.

(a) $\iint_R 6x^2y\,dA$, R=[2, 4]×[1, 2]

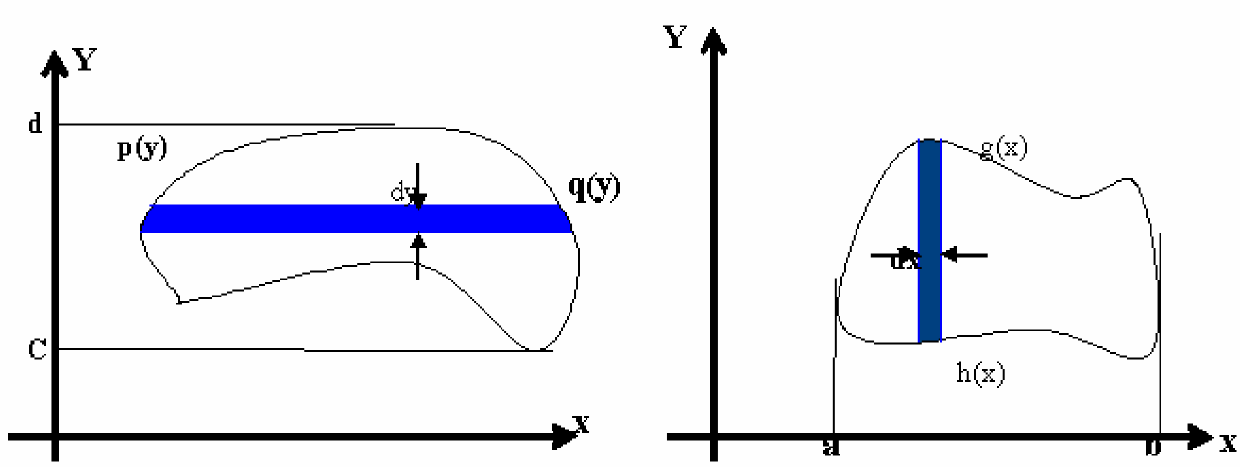
(b) $\iint_R 2x - 4y^3\,dA$, R = [−5, 4]×[0,3]

In the above example, the region R was a rectangle, so the limits on x and y were obvious. But what are the limits on x and y for a general region R? Here it matters in which order we perform the integration. Suppose we perform the integration with respect to x first, as in (4.1). Then we should express the region R in the form

$$R: c \le y \le d \quad p(y) \le x \le q(y)$$

$$\iint_R f(x,y)dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx \right] dy$$ and the inner integral is $\int_{p(y)}^{q(y)} f(x,y)dx := P(y)$

For which y is just a constant, has limits which may be (and in general are) functions of y. Physically we are summing the area under the curve along thin vertical strips of width dy, of length q ï p



The outer integral now is $\int_c^d P(y)dy$ is not a function of x or y so its limits are constants. This is the sum of all the horizontal strips between y = c and y = d.

Example 1 Evaluate each of the following integrals over the given region *D*.

(a)

$$\iint_D \mathbf{e}^{\frac{x}{y}}\, dA\,,\quad D = \left\{(x,y)\,|\,1 \le y \le 2,\ y \le x \le y^3\right\}$$

(b) $\iint_D 4xy - y^3\, dA$, D is the region bounded by $y = \sqrt{x}$ and $y = x^3$.

(c) $\iint_D 6x^2 - 40y\, dA$, D is the triangle with vertices $(0,3)$, $(1,1)$, and $(5,3)$.

Example 2 Evaluate the following integrals by first reversing the order of integration.

(a) $\int_0^3 \int_{x^2}^9 x^3 \mathbf{e}^{y^3}\, dy\, dx$

(b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}\, dx\, dy$

Properties of double integrals

The following are more or less obvious from thinking of the integral as the volume under the height function f (x, y). If limits are constant (the region is rectangle) and f(x,y)= h(x)g(y) then

(i) $$\int_a^b \int_c^d h(x)g(y)dxdy = \int_a^b h(x)dx \int_c^d g(y)dy$$

(ii) $$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$

Application of Double Integrals

Given a distribution of mass in a region R of the (x,y) plane:

- **Area** $A = \iint_R dxdy$
- **Mass** $M = \iint_R \rho(x,y)dxdy$ where $\rho(x,y)$ is the density.
- The **Centre of Gravity** of the mass in R has co-ordinates $\bar{x}, \bar{y}$ where

  $\bar{x} = \frac{1}{M}\iint_R x\rho(x,y)dxdy$, $\bar{y} = \frac{1}{M}\iint_R y\rho(x,y)dxdy$

- The **Moment of Inertia** of the mass R about the x and y axes respectively.

$$I_x = \iint_R y^2\rho(x,y)dxdy \quad I_y = \iint_R x^2\rho(x,y)dxdy.$$

Example:

Let f(x,y)=1 be the density over the quarter disc $R: x^2 + y^2 \leq 1$ in the first quadrant. How to find the bounds of integration?

Note polar co-ordinates are most useful when the region R is composed of segments of circles and straight lines. Of course, we have to calculate the new limits on the r and θ integrals carefully.

Example 1 Evaluate the following integrals by converting them into polar coordinates.

(a) $\iint_D 2xy\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

(b) $\iint_D \mathbf{e}^{x^2+y^2}\,dA$, D is the unit circle centered at the origin.

More generally, if we change co-ordinates to some other co-ordinate system

x = (u,v), y = (u,v)

Then it can be shown that the infinitesimal piece of area

$dA = dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$, $J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| := \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the determinant of the so called **Jacobian Matrix**. Taking the case of polar co-ordinates

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|drd\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}drd\theta \quad = rdrd\theta$$

as derived earlier from the first principles. We shall return to use of Jacobian matrices for changing co-ordinates in lecture 7.