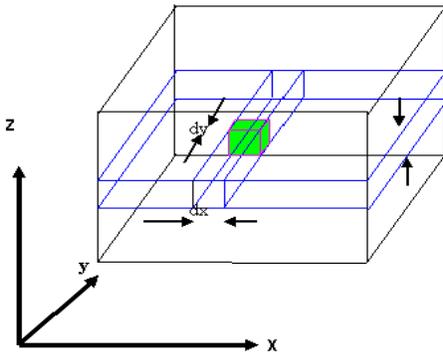


Triple Integrals

In principle triple integral are the same as double integrals, with an extra unknown. Triple integrals are also known as **Volume Integrals**. Consider a solid body occupying a volume V in 3D. Let its density by $\rho(x, y, z)$. The mass of a small element $\Delta x \Delta y \Delta z$ is $\rho(x, y, z) \Delta x \Delta y \Delta z$



Hence the total mass of the solid is $M = \iiint_V \rho(x, y, z) dx dy dz$ best illustrated by example

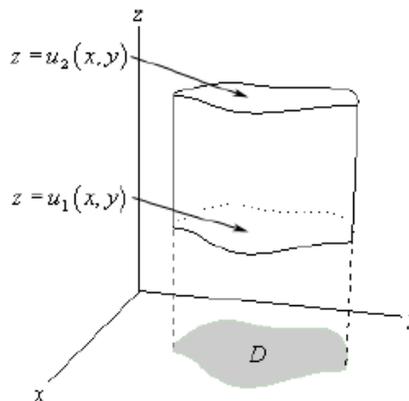
Example 1 Evaluate the following integral.

$$\iiint_B 8xyz \, dV, \quad B = [2, 3] \times [1, 2] \times [0, 1]$$

Before moving on to more general regions let's get a nice geometric interpretation about the triple integral out of the way so we can use it in some of the examples to follow.

Fact : The volume of the three-dimensional region E is given by the integral, $V = \iiint_E dV$

Let's now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

In this case we define the region E as follows,

where $(x, y) \in D$ is the notation that means that the point (x, y) lies in the region D from the xy -plane. In this case we will evaluate the triple integral as follows,

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] dA$$

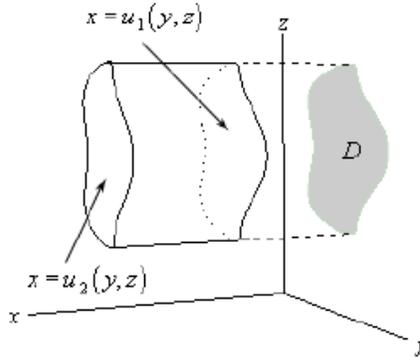
where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to x , we can integrate first

with respect to y , or we can use polar coordinates as needed.

Example:

Evaluate $\iiint_E (x^2 + y^2 + z^2) dx dy dz$ where E is the region bounded by $x + y + z = a (> 0)$, $x = 0$, $y = 0$, $z = 0$

Let's now move onto the second possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



For this possibility we define the region E as follows,

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

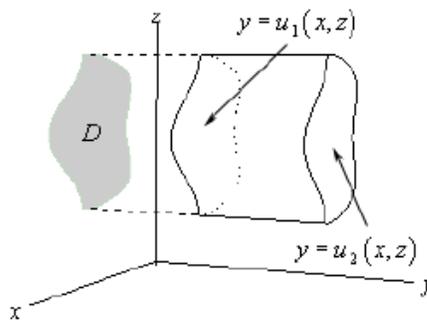
So, the region D will be a region in the yz -plane. Here is how we will evaluate these integrals.

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) dx \right] dA$$

As with the first possibility we will have two options for doing the double integral in the yz -plane as well as the option of using polar coordinates if needed.

Example 3 Determine the volume of the region that lies behind the plane $x + y + z = 8$ and in front of the region in the yz -plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

We now need to look at the third (and final) possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



In this final case E is defined as,

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

and here the region D will be a region in the xz -plane. Here is how we will evaluate these integrals.

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) dy \right] dA$$

Where we will can use either of the two possible orders for integrating D in the xz -plane or we can use polar coordinates if needed.

Triple Integrals in Cylindrical Coordinates

In this section we want do take a look at triple integrals done completely in Cylindrical Coordinates. The cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates. $x = r \cos \theta$ $y = r \sin \theta$ $z = z$

In order to do the integral in cylindrical coordinates we will need to know what dV will become in terms of cylindrical coordinates. We will be able to show in the Change of Variables section of this chapter that, $dv = rdzdrd\theta$

The region, E , over which we are integrating becomes,

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

$$= \{(r, \theta, z) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), u_1(r \cos \theta, r \sin \theta) \leq z \leq u_2(r \cos \theta, r \sin \theta)\}$$

Note that we've only given this for E 's in which D is in the xy -plane. We can modify this accordingly if D is in the yz -plane or the xz -plane as needed.

In terms of cylindrical coordinates a triple integral is,

$$\iiint_E f(x, y, z) dv = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} r f(r \cos \theta, r \sin \theta, z) dz dr d\theta$$

Don't forget to add in the r and make sure that all the x 's and y 's also get converted over into Cylindrical coordinates. Let's see an example.

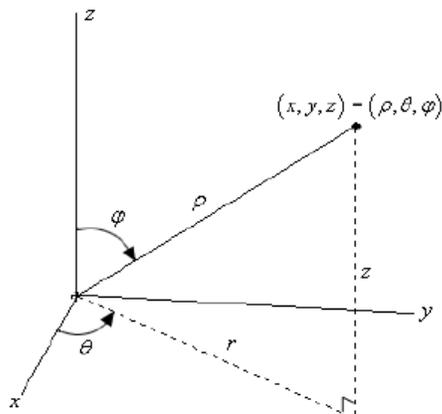
Example:

Find the volume of the region above the xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

Triple Integrals in Spherical Coordinates

In the previous section we looked at doing integrals in terms of cylindrical coordinates and we now need to take a quick look at doing integrals in terms of spherical coordinates.

First, we need to know how spherical coordinates are defined. The following sketch shows the relationship between the Cartesian and spherical coordinate systems.



Here are the conversion formulas for spherical coordinates.

$$x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

We also have the following restrictions on the coordinates.

$$\rho \geq 0 \quad 0 \leq \varphi \leq \pi$$

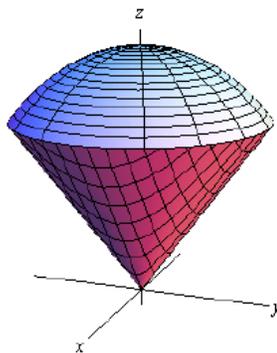
For our integrals we are going to restrict E down to a spherical wedge. This will mean that we are going to take ranges for the variables as follows,

$$a \leq \rho \leq b$$

$$\alpha \leq \theta \leq \beta$$

$$\delta \leq \varphi \leq \gamma$$

Here is a quick sketch of a spherical wedge in which the lower limit for both ρ and φ are zero for reference purposes. Most of the wedges we'll be working with will fit into this pattern.



From this sketch we can see that E is really nothing more than the intersection of a sphere and a cone. In the next section we will show that

$$dv = \rho^2 \sin \varphi d\rho d\theta d\varphi$$

Therefore the integral will become,

$$\iiint_E f(x, y, z) dv = \int_a^b \int_\alpha^\beta \int_\delta^\gamma \rho^2 \sin \varphi f(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) d\rho d\theta d\varphi$$

This looks bad, but given that the limits are all constants the integrals here tend to not be too bad.

Example: Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $z^2 \sin^2 \alpha = (x^2 + y^2) \cos^2 \alpha$ where α is a constant such that $0 \leq \alpha \leq \pi$.