Change of Variables

We had the substitution rule that told us that,

$$\int_a^b f(g(x))g'(x)dx = \int_c^d f(u)du \quad \text{Where } u = g(x)$$

In essence this is taking an integral in terms of x's and changing it into terms of u's. We want to do something similar for double and triple integrals. In fact we've already done this to certain extent when we converted double integrals to polar coordinates and when we converted triple integrals to cylindrical or spherical coordinates.

Example 1: Determine the new region that we get by applying the given transformation to the region R.

(a) R is the ellipse $x^2 + \frac{y^2}{36} = 1$ and the transformation is $x = \frac{u}{2}, y = 3v$.

(b) R is the region bounded by $y = -x + 4, y = x + 1$ and $y = \frac{x}{3} - \frac{4}{3}$ and the transformation is

$$x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v)$$

Change of variables for a double integral

Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x = g(u,v), y = h(u,v)$ the region becomes S and the integral becomes,

$$\iint_D f(x,y)dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

Example 2: Show that when changing to polar coordinates we have $dA = rdrd\theta$

So, the formula we used in the section on polar integrals was correct.
Now, let's do a couple of integrals.

Example 3: Evaluate $\iint_R (x+y)dA$ where R is the trapezoidal region with vertices given by $(0,0), (5,0), \left(\frac{5}{2}, \frac{5}{2}\right) and \left(\frac{5}{2}, \frac{-5}{2}\right)$ Using the transformation $x = 2u + 3v, y = 2u - 3v$.

Let's now briefly look at triple integrals. In this case we will again start with a region R and use the transformation $x = g(u,v,w)$, $y=h(u,v,w)$, and $z=k(u,v,w)$ to transform the region into the new region S.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

In this case the Jacobian is defined in terms of the determinant of a 3x3 matrix. We saw how to evaluate these when we looked at cross products .The integral under this transformation is,

$$\iiint_R f(x,y,z)dV = \int\iint_S f(g(u,v,w),h(u,v,w),k(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$$

As with double integrals we can look at just the differentials and note that we must have

$$dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|dudvdw$$

We're not going to do any integrals here, but let's verify the formula for dV for spherical coordinates.

Example 4: Verify that $dV = \rho^2 \sin\varphi d\rho d\theta d\varphi$ when using spherical coordinates.

Surface Integrals

Introduction

In the previous chapter we looked at evaluating integrals of functions or vector fields where the points came from a curve in two- or three-dimensional space. We now want to extend this idea and integrate functions and vector fields where the points come from a surface in three dimensional space. These integrals are called surface integrals.

Parametric Surfaces

With surfaces we'll take points, (u,v) , out of some two dimensional space D and plug them into $\underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}$ and the resulting set of vectors will be the position vectors for the points on the surface S that we are trying to parameterize. This is often called the **parametric representation** of the **parametric surface** S.

We will sometimes need to write the **parametric equations** for a surface. There are really nothing more than the components of the parametric representation explicitly written down.

$x = x(u,v), y = y(u,v), z = z(u,v)$

Example 1*:* Determine the surface given by the parametric representation

$\underline{r}(u,v) = u\underline{i} + u\cos v\,\underline{j} + u\sin v\underline{k}$

We are much more likely to need to be able to write down the parametric equations of a surface than identify the surface from the parametric representation so let's take a look at some examples of this.

Example 2: Give parametric representations for each of the following surfaces.

(a) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$

(b) The elliptic paraboloid $x = 5y^2 + 2z^2 - 10$ that is in front of the yz-plane.

Now that we have practice writing down some parametric representations for some surfaces let's take a quick look at a couple of applications. Let's take a look at finding the tangent plane to the parametric surface S given by, $\underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}$

First define,

$$\underline{r}_u(u,v) = \frac{\partial x}{\partial u}(u,v)\underline{i} + \frac{\partial y}{\partial u}(u,v)\underline{j} + \frac{\partial z}{\partial u}(u,v)\underline{k}$$

$$\underline{r}_v(u,v)=\frac{\partial x}{\partial v}(u,v)\underline{i}+\frac{\partial y}{\partial v}(u,v)\underline{j}+\frac{\partial z}{\partial v}(u,v)\underline{k}$$

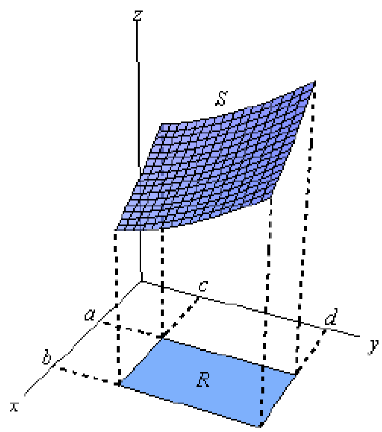
Now, provided $\underline{r}_u \times \underline{r}_v \neq 0$ it can be shown that the vector $\underline{r}_u \times \underline{r}_v$ will be orthogonal to the surface *S*. This means that it can be used for the normal vector that we need in order to write down the equation of a tangent plane. This is an important idea that will be used many times throughout the next couple of sections.

Let's take a look at an example.

Example 3 Find the equation of the tangent plane to the surface given by $\underline{r}(u,v)=u\underline{i}+2v^2\,\underline{j}+(u^2+v)\underline{k}$ at the point (2, 2,3).

Surface Integrals

It is now time to think about integrating functions over some surface, *S*, in three-dimensional space. Let's start off with a sketch of the surface *S* since the notation can get a little confusing once we get into it. Here is a sketch of some surface *S*.



The region *S* will lie above (in this case) some region *D* that lies in the *xy*-plane. Now, how we evaluate the surface integral will depend upon how the surface is given to us.
There are essentially two separate methods here, although as we will see they are really the same.
First, let's look at the surface integral in which the surface *S* is given by $z=g(x,y)$. In this case the surface integral is,

$$\iint_S f(x,y,z)ds=\iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}dA$$

The second method for evaluating a surface integral is for those surfaces that are given by the parameterization, $\underline{r}(u,v)=x(u,v)\underline{i}+y(u,v)\underline{j}+z(u,v)\underline{k}$

In these cases the surface integral is,
$\iint_S f(x,y,z)ds=\iint_D f(\underline{r}(u,v))\left\|\underline{r}_u\times\underline{r}_v\right\|dA$ where *D* is the range of the parameters that trace out the surface *S*.
Before we work some examples let's notice that since we can parameterize a surface given by $z=g(x,y)$ as, $\underline{r}(x,y)=x\underline{i}+y\underline{j}+g(x,y)\underline{k}$
We can always use this form for these kinds of surfaces as well. In fact it can be shown that, $\left\|\underline{r}_x\times\underline{r}_y\right\|=\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}$

For these kinds of surfaces. You might want to verify this for the practice of computing these cross products. Let's work some examples.

Example : Evaluate $\iint_S U(x,y,z)ds$ where S is the surface of the paraboloid $z = 2-(x^2+y^2)$, above the xy plane and $U(x,y,z)$ is equal to (a) 1 (b) (x^2+y^2) .Give a physical interpretation in each case.

Tutorial

1). Give parametric representations for each of the following surfaces

(a) The sphere $x^2+y^2+z^2=30$

sol: $\underline{r}(\theta,\varphi)=\sqrt{30}\sin\varphi\cos\theta\underline{i}+\sqrt{30}\sin\varphi\sin\theta\underline{j}+\sqrt{30}\cos\varphi\underline{k}$

(b) The cylinder $y^2+z^2=25$

Sol: $\underline{r}(x,\theta)=x\underline{i}+5\sin\theta\underline{j}+5\cos\theta\underline{k}$.

2). Evaluate $\iint_S yds$ where S is the portion of the cylinder $x^2+y^2=3$ that lies between $z=0$ and $z=6$. Sol: 0

3). $\iint_S (y+z)ds$ where S is the surface whose side is the cylinder $x^2+y^2=3$, whose bottom is the disk $x^2+y^2\le 3$ in the xy plane and whose top is the plane $z=4-y$.

Sol: $\frac{\pi}{2}\left(29\sqrt{3}+24\sqrt{2}\right)$

4). Evaluate the double integral:

(a) $\int_3^4\int_1^2 \frac{x}{x-y}\,dy\,dx$ (b) $\int_2^3\int_{-1}^2 \frac{1}{(x+y)^2}\,dy\,dx$

5). Find the volume of the solid region bounded below by the given rectangle in the -plane and above by the graph of the given surface.

(a) $f(x,y)=\frac{xy}{\sqrt{x^2+y^2+1}}$ on $0\le x\le 1, 0\le y\le 1$

(b) $f(x,y)=(x+y)^5$ on $0\le x\le 1,\ 0\le y\le 1$

(*6)* If f is a constant function, say $f(x,y)=k,$ and $R=[a,b]\times[c,d],$ show that

$$\iint_R k\,dA = k(b-a)(d-c)$$

Example 3 Evaluate $\iint_S y\,dS$ where *S* is the portion of the cylinder $x^2+y^2=3$ that lies between $z=0$ and $z=6$.

Solution
We parameterized up a cylinder in the previous section. Here is the parameterization of this cylinder.

$$\vec{r}(z,\theta)=\sqrt{3}\cos\theta\,\vec{i}+\sqrt{3}\sin\theta\,\vec{j}+z\,\vec{k}$$

The ranges of the parameters are,

$$0\le z\le 6 \qquad 0\le\theta\le 2\pi$$

Now we need $\vec{r}_z\times\vec{r}_\theta$. Here are the two vectors.

$$\vec{r}_z(z,\theta)=\vec{k}$$
$$\vec{r}_\theta(z,\theta)=-\sqrt{3}\sin\theta\,\vec{i}+\sqrt{3}\cos\theta\,\vec{j}$$

Here is the cross product.

$$\vec{r}_z\times\vec{r}_\theta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & 1\\ -\sqrt{3}\sin\theta & \sqrt{3}\cos\theta & 0\end{vmatrix}$$
$$=-\sqrt{3}\cos\theta\,\vec{i}-\sqrt{3}\sin\theta\,\vec{j}$$

The magnitude of this vector is,

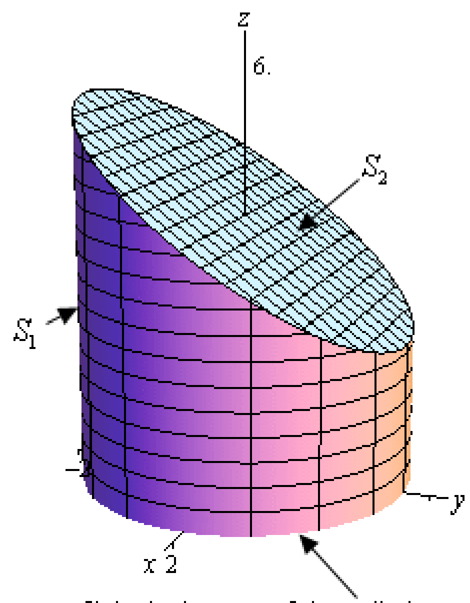
$$\left\|\vec{r}_z\times\vec{r}_\theta\right\|=\sqrt{3\cos^2\theta+3\sin^2\theta}=\sqrt{3}$$

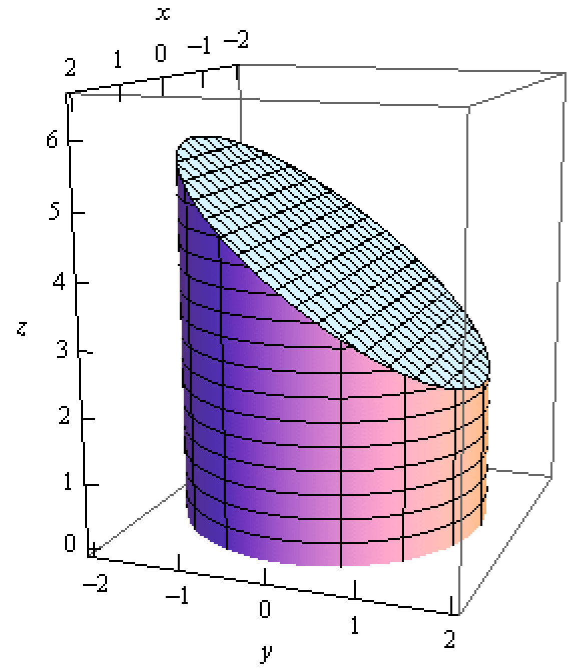
The surface integral is then,

$$\begin{aligned}\iint_S y\,dS&=\iint_D \sqrt{3}\sin\theta\left(\sqrt{3}\right)dA\\&=3\int_0^{2\pi}\int_0^6 \sin\theta\,dz\,d\theta\\&=3\int_0^{2\pi}6\sin\theta\,d\theta\\&=\left(-18\cos\theta\right)\Big|_0^{2\pi}\\&=0\end{aligned}$$

Example 4 Evaluate $\iint_S y+z\,dS$ where *S* is the surface whose side is the cylinder $x^2+y^2=3$, whose bottom is the disk $x^2+y^2\le 3$ in the *xy*-plane and whose top is the plane $z=4-y$.

Solution
There is a lot of information that we need to keep track of here. First, we are using pretty much the same surface (the integrand is different however) as the previous example. However, unlike the previous example we are putting a top and bottom on the surface this time. Let's first start out with a sketch of the surface.





Actually we need to be careful here. There is more to this sketch than the actual surface itself. We're going to let S_1 be the portion of the cylinder that goes from the *xy*-plane to the plane. In other words, the top of the cylinder will be at an angle. We'll call the portion of the plane that lies inside (*i.e.* the cap on the cylinder) S_2. Finally, the bottom of the cylinder (not shown here) is the disk of radius $\sqrt{3}$ in the *xy*-plane and is denoted by S_3.

In order to do this integral we'll need to note that just like the standard double integral, if the surface is split up into pieces we can also split up the surface integral. So, for our example we will have,

$$\iint_S y + z\,dS = \iint_{S_1} y + z\,dS + \iint_{S_2} y + z\,dS + \iint_{S_3} y + z\,dS$$

We're going to need to do three integrals here. However, we've done most of the work for the first one in the previous example so let's start with that.

S_1 : <u>The Cylinder</u>

The parameterization of the cylinder and $\left\|\vec{r}_z \times \vec{r}_\theta\right\|$ is,

$$\vec{r}\left(z,\theta\right) = \sqrt{3}\cos\theta\,\vec{i} + \sqrt{3}\sin\theta\,\vec{j} + z\,\vec{k} \qquad\qquad \left\|\vec{r}_z \times \vec{r}_\theta\right\| = \sqrt{3}$$

The difference between this problem and the previous one is the limits on the parameters. Here they are.

$$0 \le \theta \le 2\pi$$
$$0 \le z \le 4 - y = 4 - \sqrt{3}\sin\theta$$

The upper limit for the *z*'s is the plane so we can just plug that in. However, since we are on the cylinder we know what *y* is from the parameterization so we will also need to plug that in.

Here is the integral for the cylinder.

$$\iint_{S_1} y+z\,dS = \iint_D \left(\sqrt{3}\sin\theta + z\right)\left(\sqrt{3}\right)dA$$
$$= \sqrt{3}\int_0^{2\pi}\int_0^{4-\sqrt{3}\sin\theta} \sqrt{3}\sin\theta + z\,dz\,d\theta$$
$$= \sqrt{3}\int_0^{2\pi} \sqrt{3}\sin\theta\left(4-\sqrt{3}\sin\theta\right) + \frac{1}{2}\left(4-\sqrt{3}\sin\theta\right)^2 d\theta$$
$$= \sqrt{3}\int_0^{2\pi} 8 - \frac{3}{2}\sin^2\theta\,d\theta$$
$$= \sqrt{3}\int_0^{2\pi} 8 - \frac{3}{4}\left(1-\cos(2\theta)\right)d\theta$$
$$= \sqrt{3}\left(\frac{29}{4}\theta + \frac{3}{4}\sin(2\theta)\right)\Big|_0^{2\pi}$$
$$= \frac{29\sqrt{3}\pi}{2}$$

S_2 : <u>Plane on Top of the Cylinder</u>

In this case we don't need to do any parameterization since it is set up to use the formula that we gave at the start of this section. Remember that the plane is given by $z = 4 - y$. Also note that, for this surface, *D* is the disk of radius $\sqrt{3}$ centered at the origin.

Here is the integral for the plane.

$$\iint_{S_2} y+z\,dS = \iint_D (y+4-y)\sqrt{(0)^2+(-1)^2+1}\,dA$$
$$= \sqrt{2}\iint_D 4\,dA$$

Don't forget that we need to plug in for *z*! Now at this point we can proceed in one of two ways. Either we can proceed with the integral or we can recall that $\iint_D dA$ is nothing more than the area of *D* and we know that *D* is the disk of radius $\sqrt{3}$ and so there is no reason to do the integral.

Here is the remainder of the work for this problem.

$$\iint_{S_2} y+z\,dS = 4\sqrt{2}\iint_D dA$$
$$= 4\sqrt{2}\left(\pi\left(\sqrt{3}\right)^2\right)$$
$$= 12\sqrt{2}\pi$$

S_3 : <u>Bottom of the Cylinder</u>

Again, this is set up to use the initial formula we gave in this section once we realize that the equation for the bottom is given by $g(x,y)=0$ and D is the disk of radius $\sqrt{3}$ centered at the origin. Also, don't forget to plug in for z.

Here is the work for this integral.

$$\begin{aligned}
\iint_{S_3} y+z\,dS &= \iint_D (y+0)\sqrt{(0)^2+(0)^2+(1)^2}\,dA \\
&= \iint_D y\,dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{3}} r^2\sin\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{3}r^3\sin\theta\right)\Big|_0^{\sqrt{3}}\,d\theta \\
&= \int_0^{2\pi}\sqrt{3}\sin\theta\,d\theta \\
&= -\sqrt{3}\cos\theta\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

We can now get the value of the integral that we are after.

$$\begin{aligned}
\iint_S y+z\,dS &= \iint_{S_1} y+z\,dS + \iint_{S_2} y+z\,dS + \iint_{S_3} y+z\,dS \\
&= \frac{29\sqrt{3}\pi}{2} + 12\sqrt{2}\,\pi + 0 \\
&= \frac{\pi}{2}\left(29\sqrt{3}+24\sqrt{2}\right)
\end{aligned}$$