Surface Integrals of Vector Fields

In line integrals the orientation of the curve we were integrating along could change the answer. The same thing will hold true with surface integrals. So, before we get into doing surface integrals of vector fields we first need to introduce the idea of an oriented surface. Let's start with a surface that has two sides, that has a tangent plane at every point (except possibly along the boundary). i.e. every point will have two unit normal vectors, $\underline{n}_1$ and $\underline{n}_2 = -\underline{n}_1$. The set that we choose will give the surface an orientation. First we need to define a closed surface.

We say that the closed surface S has a positive orientation if we choose the set of unit normal vectors that point outward from the region E while the negative orientation will be the set of unit normal vectors that point in towards the region E. Note that this convention is only used for closed surfaces.

First, let's suppose that the function is given by $z = g(x, y)$. The function, $f(x, y, z) = z - g(x, y)$
The surface is then given by the equation $f(x, y, z) = 0$.

∇f will be orthogonal (or normal) to the surface . So the unit vector is, $\underline{n} = \dfrac{\nabla f}{\|\nabla f\|}$

Then $\underline{n} = \dfrac{\nabla f}{\|\nabla f\|} = \dfrac{-g_x \underline{i} - g_y \underline{j} + \underline{k}}{\sqrt{(g_x)^2 + (g_y)^2 + 1}}$

Now, we need to discuss how to find the unit normal vector if the surface is given parametrically as,
$\underline{r}(u, v) = x(u, v)\underline{i} + y(u, v)\underline{j} + z(u, v)\underline{k}$

In this case recall that the vector $\underline{r}_u \times \underline{r}_v$ will be normal to the tangent plane at a particular point. But if the vector is normal to the tangent plane at a point then it will also be normal to the surface at that point. So, this is a normal vector. So, in the case of parametric surfaces one of the unit normal vectors will be,

$$\underline{n} = \frac{\underline{r}_u \times \underline{r}_v}{\|\underline{r}_u \times \underline{r}_v\|}$$

Given a vector field $\underline{F}$ with unit normal vector $\underline{n}$ then the surface integral of $\underline{F}$ over the surface S is given by,

$$\iint_S \underline{F}.d\underline{S} = \iint_S \underline{F}.\underline{n}dS$$

Where the right hand integral is a standard surface integral. This is sometimes called the flux of $\underline{F}$ across S.
Let's first start by assuming that the surface is given by $z = g(x, y)$. In this case let's also assume that the vector field is given by $\underline{F} = P\underline{i} + Q\underline{j} + R\underline{k}$ that the orientation that we are after is the "upwards" orientation. Under all of these assumptions the surface integral of $\underline{F}$ over S is,

$$\iint_S \underline{F}.d\underline{S} = \iint_S \underline{F}.\underline{n}dS$$

Now, remember that this assumed the "upward" orientation. If we'd needed the "downward" orientation then we would need to change the signs on the normal vector. Notice as well that because we are using the unit normal vector the messy square root will always drop out. Note that the square root is nothing more than,

$$\sqrt{(g_x)^2 + (g_y)^2 + 1} = \|\nabla f\|$$

So in the following work we will probably just use this notation in place of the square root when we can to make things a little simpler.

Let's now take a quick look at the formula for the surface integral when the surface is given parametrically by $\underline{r}(u, v)$ In this case the surface integral is,

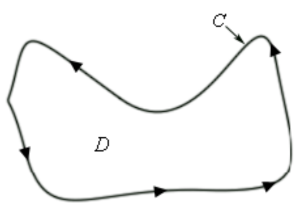
$$\iint_S \underline{F}.d\underline{S} = \iint_S \underline{F}.\underline{n}dS$$

Again note that we may have to change the sign on $\underline{r}_u \times \underline{r}_v$ to match the orientation of the surface and so there is once again really two formulas here. Also note that again the magnitude cancels in this case and so we won't need to worry that in these problems either.
Note as well that there are even times when we will used the definition,
$\iint_S \underline{F}.d\underline{S} = \iint_S \underline{F}.\underline{n}dS$ directly. We will see an example of this below.

Example 1: Evaluate $\iint_S \underline{F}.d\underline{S}$ where $\underline{F} = y\underline{j} - z\underline{k}$ and S is the surface given by the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ at y =1. Assume that S has positive orientation.

Green's Theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals.
Let's start off with a simple closed curve *C* and let *D* be the region enclosed by the curve. Here is a sketch of such a curve and region.



First, notice that because the curve is simple and closed there are no holes in the region *D*. Also notice that a direction has been put on the curve. We will use the convention here that the curve *C* has a **positive orientation** if it is traced out in a counter-clockwise direction. As we traverse the path following the positive orientation the region *D* must always be on the left. Given curves/regions such as this we have the following theorem.

Green's Theorem

Let *C* be a positively oriented, piecewise smooth, simple, closed curve and let *D* be the region enclosed by the curve. If *P* and *Q* have continuous first order partial derivatives on *D* then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

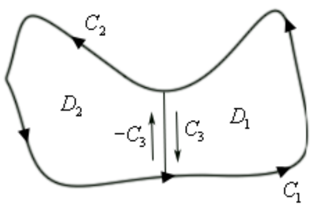
$$\oint_C P\,dx + Q\,dy$$

This notation do assume that *C* satisfies the conditions of Green's Theorem so be careful in using them.
Also, sometimes the curve *C* is not thought of as a separate curve but instead as the boundary of some region *D* and in these cases you may see *C* denoted as ∂D.
Let's work a couple of examples.

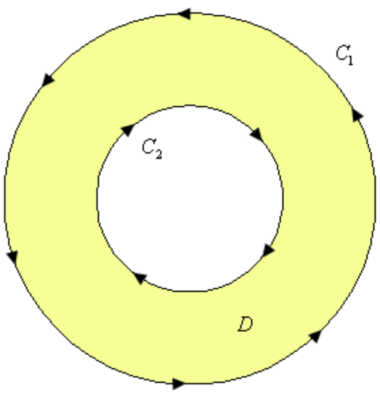
Example 1: Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2 y^3 dy$ where *C* is the triangle with vertices $(0,0)$, $(1, 0)$, $(1, 2)$ with positive orientation.

Example 2*:* Evaluate $\oint_C y^3 dx - x^3 dy$ where C is the positively oriented circle of radius 2 centered at the origin.

So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions.



What this exercise has shown us is that if we break a region up as we did above then the portion of the line integral on the pieces of the curve that are in the middle of the region (each of which are in the opposite direction) will cancel out. This idea will help us in dealing with regions that have holes in them. To see this let's look at a ring.



Notice that both of the curves are oriented positively since the region D is on the left side as we traverse the curve in the indicated direction. Note as well that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work and we need to resort to the second definition that we gave above.

Example 3*:* Evaluate $\oint_C y^3 dx - x^3 dy$ where C are the two circles of radius 2 and radius 1centered at the origin with positive orientation.

We will close out this section with an interesting application of Green's Theorem. Recall that we can determine the area of a region D with the following double integral.

$$A = \iint_D dA$$

Let's think of this double integral as the result of using Green's Theorem. In other words, let's assume that $Q_x - P_y = 1$ and see if we can get some functions P and Q that will satisfy this. There are many functions that will satisfy this. Here are some of the more common functions.

$$P = 0 \quad P = -y \quad P = \frac{y}{2}$$
$$Q = x \quad Q = 0 \quad Q = \frac{x}{2}$$

Then, if we use Green's Theorem in reverse we see that the area of the region D can also be computed by evaluating any of the following line integrals.

$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$ where C is the boundary of the region D.

Tutorial

1) **:** Evaluate $\iint_S \underline{F}.d\underline{S}$ where $\underline{F} = x\underline{i} + y\underline{j} + z^4\underline{k}$ and S is the upper half of the sphere $x^2 + y^2 + z^2 = 9$, and the disk $x^2 + y^2 \leq 9$ in the plane $z = 0$. Assume that S has positive orientation.

Sol: 279π

2) Use Green's Theorem to find the area of a disk of radius a.
Sol: πa^2

3) Use Green's theorem to evaluate the line integral $\oint_C 4y\,dx - 3x\,dy$ around the curve C defined by the ellipse $2x^2 + y^2 = 4$ oriented counterclockwise
Sol: $-14\sqrt{2}\pi$

4) Use Green's theorem to evaluate the line integral $\oint_C 4xy\,dx$ around the curve C defined by the unit circle oriented clockwise.
Sol: 0

5) Use Green's theorem to evaluate the line integral $\oint_C y^2 dx + x\,dy$ around the curve C defined by the square with vertices (0,0) (2,0) (2,2) (0,2) oriented counterclockwise.
Sol: -4