Example: Use n equal partitions of $[0,1]$ to estimate the "area" under the curve $f(x) = x^2$ using

1. left corner of the intervals
2. right corner of the intervals
3. midpoint of the interval
4. line joining the left and right corners of the interval

Definitions:

P is a **partition** of $[a, b]$ iff it is a set of the form $P = \{a = x_0, x_1, \cdots, x_n = b\}$
P^* is a **refinement** of P iff $P^* \supseteq P$
P is a **common refinement** of P_1, P_2 iff $P = P_1 \cup P_2$
$\mathcal{P}[a, b]$ is the set of all partitions of $[a, b]$

Definition: Upper and Lower Reimann Sums $f: [a, b] \to \mathbb{R}$ is a bounded function
$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ where $M_i = \sup\{f(x) \mid x_{i-1} \le x \le x_i\}$
$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ where $m_i = \inf\{f(x) \mid x_{i-1} \le x \le x_i\}$

Definition: **Upper and Lower Reimann Integrals**
$\overline{\int_a^b f(x)dx} = \inf\{U(P, f) \mid P \in \mathcal{P}[a, b]\}$
$\underline{\int_a^b f(x)dx} = \sup\{L(P, f) \mid P \in \mathcal{P}[a, b]\}$

Definition:

f is **Reimann Integrable** on $[a, b]$ or $f \in \Re[a, b]$ iff $\underline{\int_a^b f(x)dx} = \overline{\int_a^b f(x)dx}$

Reimann Integral of f is the common value denoted by $\int_a^b f(x)dx$

Theorem: P^* is a refinement of P

1. $L(P, f) \le L(P^*, f)$
2. $U(P^*, f) \le U(P, f)$

Theorem: $\underline{\int_a^b f(x)dx} \le \overline{\int_a^b f(x)dx}$

Theorem: $f \in \mathcal{R}[a, b]$ iff $\forall \varepsilon > 0\ \exists P \in \mathcal{P}[a, b];\ U(P, f) - L(P, f) < \varepsilon$

Theorem: If $f \in \mathcal{R}[a, b]$ and $P \in \mathcal{P}[a, b]$ such that $t_i \in [x_{i-1}, x_i]$ then
$U(P, f) - L(P, f) < \varepsilon \Rightarrow \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < \varepsilon$

Theorem: $f \in \mathcal{C}[a, b] \Rightarrow f \in \mathcal{R}[a, b]$

Theorems: $f, g \in \mathcal{R}[a, b]$

1. $f + g \in \mathcal{R}[a, b]$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $fg \in \mathcal{R}[a, b]$
3. $|f| \in \mathcal{R}[a, b]$ and $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$
4. $f \le g \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$
5. $f \le M \Rightarrow \int_a^b f(x)dx \le M(b - a)$
6. $c \in [a, b] \Rightarrow f \in \mathcal{R}[a, c], f \in \mathcal{R}[c, b]$ and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Theorem: Fundamental Theorem of Calculus

If $f \in \mathcal{R}[a, b]$ and there is a differentiable function F such that $F' = f$ then
$\int_a^b f(x)dx = F(b) - F(a)$

Theorem: Second Fundamental Theorem of Calculus

If $f \in \mathcal{R}[a, b]$ and $x \in [a, b]$ and $F(x) = \int_a^x f(x)dx$ then

1. F is continuous on $[a, b]$.
2. If f is continuous at a point $x_0 \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Theorem: Integration by Parts

F, G differentiable on $[a, b]$, $F' = f \in \mathcal{R}[a, b]$ and $G' = f \in \mathcal{R}[a, b]$ *then*

$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$

Theorem: Change of Variable

g has continuous derivative g' *on* $[c, d]$. f is continous on $g([c, d])$ and let $F(x) = \int_{g(c)}^{x} f(t)dt, x \in g([c, d])$. Then for each $x \in [c, d]$, $\int_c^x f(g(t))g'(t)dt$ exixts and has value $F(g(x))$.

Theorem: Mean Value Theorem for Integrals

$f \in \mathcal{R}[a, b]$ with $m \leq f \leq M$. Then $\exists\ c \in (m, M)$ such that $\int_a^b f(x)dx = c(b - a)$.

If also $f \in \mathcal{C}[a, b]$ then $\exists\ x_0 \in (a, b)$ such that $\int_a^b f(x)dx = f(x_0)(b - a)$.

Definition: Improper Integrals of the first kind

Suppose $\int_a^b f(x)dx$ exixts for each $b \geq a$.

If $\lim_{b\to\infty} \int_a^b f(x)dx$ exists and equal to $I \in \mathbb{R}$ we say that $\int_a^\infty f(x)dx$ converges and has value I

Otherwise we say that $\int_a^\infty f(x)dx$ diverges

Definition: Improper Integrals

$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx,\ c \in \mathbb{R}$

$\int_{a^+}^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$

$\int_a^{b^-} f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$

$\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)dx, c \in (a, b)$

Example: Find $\int_{-1}^1 \frac{1}{x^2}dx$ if it exists

Theorem: Comparison Test

Assume that the proper integral $\int_a^b f(x)dx$ exists for each $b \geq a$ and suppose that $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then $\int_a^\infty g(x)dx$ converges $\Rightarrow$ $\int_a^\infty f(x)dx$ converges

Theorem: Limit Comparison Test

Assume both proper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exist for each $b \geq a$, where $f(x) \geq 0$ and $g(x) > 0$

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \text{c}$, then

1. $\text{c} \neq 0, \infty \Rightarrow \int_a^\infty f(x)dx$ converges $\Leftrightarrow$ $\int_a^\infty g(x)dx$ converges
2. $\text{c} = 0$ and $\int_a^\infty g(x)dx$ converges $\Rightarrow$ $\int_a^\infty f(x)dx$ converges
3. $\text{c} = \infty$ and $\int_a^\infty g(x)dx$ diverges $\Rightarrow$ $\int_a^\infty f(x)dx$ diverges

Note: There are similar comparison tests for other improper integrals

Example: **Gamma Function** is defined by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. Show that

1. $\Gamma(x)$ exists for all $x > 0$
2. $\Gamma(x) = (x - 1)\Gamma(x - 1)$
3. $\Gamma(n) = (n - 1)!$ for integer $n \geq 1$
4. we can use 2. to define $\Gamma(x)$ for $x < 0$
5. $\Gamma(x)$ does not exist for $x = 0, -1, -2, -3, ...$
6. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ and $\int_0^\infty e^{-t^2}dt = \sqrt{\pi}$ using $\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin \pi x}$
7. Use the formula for the the n dimesional ball $V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n$ to find volumes of 2,3,4,5 dimesional balls
8. Use the fact that $\Gamma(x + 1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$ asmptotically as $t \to \infty$ to find 10! approximately
9. What is $-\Gamma'(1)$?. It is called the Euler Constant γ and no one knows if it is rational or irrational!

Definition: nth Order Ordinary Differential Equation
$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y, x\right) = 0$

Definition: 1st Order Ordinary Differential Equation
$\frac{dy}{dx} = f(x, y)$

Example:
1. $y' - 3y^{2/3} = 0, y(0) = 0$ has two solutions $y = 0$ and $y = x^3$.
2. $y' - y^{-1} = 0, y(0) = 0$ has no solution for $x > 0$.

Definition: $f(x, y)$ is Lipschitz continuous on $R \subset \mathbb{R}^2$
$\exists L > 0, \forall (x, y_1), (x, y_2) \in R;\ |f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|$

Theorem: If $f(x, y)$ is Lipschitz continuous on $R \subset \mathbb{R}^2$ and $(x_0, y_0) \in R$ then the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ has a unique solution in R

Definition/Theorem: variable separable 1st order ODE
$f(x, y) = \frac{g(x)}{h(y)}$
$\int h(y)dy = \int g(x)dx$

Definition/Theorem: homogeneous 1st order ODE
$f(x, Vx) = g(V)$
$\frac{dy}{dx} = V + x\frac{dV}{dx} = g(V) \Rightarrow \frac{dV}{dx} = \frac{g(V) - V}{x}$: varaibale separable

Definition/Theorem: Linear 1st order ODE
$f(x, y) = Q(x) - P(x)y$
Integrating Factor: $I(x) = e^{\int P(x)dx}$
$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow I(x)\frac{dy}{dx} + I(x)P(x)y = Q(x)I(x)$
$\Rightarrow \frac{d}{dx}(I(x)y) = I(x)Q(x) \Rightarrow y = \frac{1}{I(x)}\int I(x)Q(x)dx$

Definition/Theorem: Bernoulli 1st order ODE
$f(x, y) = Q(x) - P(x)y^n$
$z = y^{1-n} \Rightarrow \frac{dz}{dx} = (1-n)y^{-n} \Rightarrow \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$: Linear

Example: Solve the first order ODEs
1. $\frac{dy}{dx} = ye^x$
2. $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$
3. $\frac{dy}{dx} - \frac{y}{x} = \ln x$

Definition: 2nd Order Linear Ordinary Differential Equation
$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$

Definition: 2nd Order Linear Ordinary Differential Equation with constant coefficients
$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$
If $\alpha^2 + p\alpha + q = (\alpha - a)(\alpha - b) = \alpha(\alpha - b) - a(\alpha - b) = \alpha(\alpha - a) - b(\alpha - a)$
$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = \frac{d}{dx}\left(\frac{dy}{dx} - ay\right) - b\left(\frac{dy}{dx} - ay\right) = \frac{dz}{dx} - bz = r(x)$ and $\frac{dy}{dx} - ay = z$: Linear 1st ordere ODEs

Example: Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$ as two Linear 1st order ODEs

Definition/Theorem:

$x^2\frac{d^2y}{dx^2}+xp\frac{dy}{dx}+qy=r(x)$

$x=e^z \Rightarrow x\frac{dy}{dx}=\frac{dy}{dz} \Rightarrow x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d^2y}{dz^2}\frac{1}{x} \Rightarrow x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$

$\frac{d^2y}{dz^2}-\frac{dy}{dx}+p\frac{dy}{dz}+qy=r(e^z) \Rightarrow \frac{d^2y}{dz^2}+(p-1)\frac{dy}{dz}+qy=r(e^z)\text{: 2nd Order Linear}$

Definition: The solutions to $\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$ (homogeneous equation)can be obtained by substituting $y=ce^{\alpha x} \Rightarrow \alpha^2+p\alpha+q=(\alpha-a)(\alpha-b)=0, \text{charasteristic equation}$

1. $a,b\in\mathbb{R}, a\neq b \Rightarrow y=ce^{ax}+de^{bx}$
2. $a,b\in\mathbb{R}, a=b \Rightarrow y=ce^{ax}+dxe^{bx}$
3. $a,b\in\mathbb{C} \Rightarrow a=a_1+ia_2=\bar{b} \Rightarrow y=ce^{a_1x}\sin a_2x+ce^{a_1x}\cos a_2x$

Definition/Theorem: The solutions to $\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x), y(x_0)=y_0, y'(x_0)=y'_0$

1. Exists and unique on an interval (c,d) where $x_0\in(c,d)\subseteq(a,b)$ and $p(x),q(x),r(x)$ continuous on (a,b).
2. The solution can be expressed as $y=y_c+y_p$
3. $y_c=au(x)+bv(x)$ (complimentary/fundamental solution) is the solution when $r(x)\equiv 0$ (homogeneous equation)and u,v (fundamental set of solutions)are linearly independent($\forall x(au(x)+bv(x)=0)\Rightarrow a=b=0$).
4. y_p (particular solution) is a solution when $r(x)\not\equiv 0$

Example: Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=\sin x$ by separately finding y_c and y_p

Definition: Wronskian

$W(u,v)(x)=uv'-vu'=\begin{vmatrix}u & v\\ u' & v'\end{vmatrix}$

Theorem: If u,v are solutions to the homogeneous equation, the Wronskian satisfies

1. $W'+p(x)W=0$
2. $W(u,v)(x)=W(u,v)(x_0)\exp\left(-\int_{x_0}^{x}p(t)dt\right)$
3. $W(u,v)(x)\neq 0$ or $W(u,v)(x)\equiv 0$
4. $W(u,v)(x_0)\neq 0 \Leftrightarrow u,v$ are linearly independent

Example: If e^{ax} is a solution to $\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+a^2y=0$ find the other independent solution.

Theorem: If u,v are fundamental solutions to the homogeneous equation, then the particular solution is given by

$y_p(x)=c(x)u(x)+d(x)v(x)$

$\begin{pmatrix}u & v\\ u' & v'\end{pmatrix}\begin{pmatrix}c'\\ d'\end{pmatrix}=\begin{pmatrix}0\\ r\end{pmatrix};\ c'=\frac{-rv}{W}=\frac{W_1}{W}, W_1=\begin{vmatrix}0 & v\\ r & v'\end{vmatrix};\ d'=\frac{ru}{W}=\frac{W_2}{W}, W_2=\begin{vmatrix}u & 0\\ u' & r\end{vmatrix}$

$y_p(x)=\int_{x_0}^{x}\frac{v(x)u(t)-u(x)v(t)}{W(u,v)(t)}r(t)dt$

Example: Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=\sin x$ using the Wronskian.

Example: Airy Equation: $\frac{d^2y}{dx^2}-xy=0$. The fundamental solutions are

1. Airy function of the first kind $\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^{\infty}\cos\left(\frac{t^3}{3}+xt\right)dt$: bounded solution as $x\to\infty$
2. Airy function of the second kind $\mathrm{Bi}(x)$: unbounded solution as $x\to\infty$

Example: Bessel Equation: $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$, n is an integer. The fundamental solutions are

1. Bessel function of the first kind $\mathrm{J_n}(x)$: bounded solution at $x=0$
2. Bessel function of the second kind $\mathrm{K_n}(x)$: unbounded solution at $x=0$

Definition: Function of two variables $f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$

Example: Draw the graphs of the following functions
1. $f(x, y) = x^2 + y^2$
2. $f(x, y) = \sqrt{x^2 + y^2}$

Definition: $\lim_{(x,y)\to(a,b)} f(x, y) = L \Leftrightarrow$
$\forall \varepsilon > 0 \exists \delta > 0, d\big((x, y), (a, b)\big) < \delta \Rightarrow |f(x, y) - L| < \varepsilon$

Note: $d\big((x, y), (a, b)\big) < \delta$ is a region around (a, b). Some options for $d\big((x, y), (a, b)\big)$ are
1. $\sqrt{(x - a)^2 + (y - b)^2}$
2. $|x - a| + |y - b|$
3. $\max\{|x - a|, |y - b|\}$

We will use the 1st option. One can show that all are equivalent, what is needed is a region around (a, b).

Example: Use the definition to show that $\lim_{(x,y)\to(2,3)} xy = 6$.

Example: Investigate the existence of the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ for the following functions
1. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$
2. $f(x, y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$
3. $f(x, y) = \begin{cases} x\sin\frac{1}{y} & , (x, y) \neq (0,0) \\ 0 & , (x, y) = (0,0) \end{cases}$

Theorem: If $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{x\to a} f(x, y)$, $\lim_{y\to b} f(x, y)$ exists then
$\lim_{y\to b} \lim_{x\to a} f(x, y) = \lim_{x\to a} \lim_{y\to b} f(x, y) = L$

Definition: Partial derivatives

$f_1(a, b) = f_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x,b)-f(a,b)}{\Delta x}$

$f_2(a, b) = f_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{\Delta y\to 0} \frac{f(a,b+\Delta y)-f(a,b)}{\Delta y}$

Definition: $f \in \mathcal{C}^1 \Leftrightarrow f_1 \in \mathcal{C}$ and $f_2 \in \mathcal{C}$

Theorem: Mean Value
1. $f \in \mathcal{C}^1$ in D
2. The circle $(x - a)^2 + (y - b)^2 \leq \delta^2$ lies inside D

$\Rightarrow$
1. $f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a + \theta_1 \Delta x, b)\Delta x + f_2(a + \Delta x, b + \theta_2 \Delta y)\Delta y$
2. $\Delta x^2 + \Delta y^2 < \delta^2$ and $0 < \theta_1, \theta_2 < 1$

Definition: Differentiability of f $(f \in \mathcal{D})$ at (a, b)
1. f_1 and f_2 exists at (a, b)
2. $f(a + \Delta x, b + \Delta y) - f(a, b) = f_1(a, b)\Delta x + f_2(a, b)\Delta y + \phi(\Delta x, \Delta y)\Delta x + \psi(\Delta x, \Delta y)\Delta y$
3. $\lim_{(\Delta x,\Delta y)\to(0,0)} \phi(\Delta x, \Delta y) = \lim_{(\Delta x,\Delta y)\to(0,0)} \psi(\Delta x, \Delta y) = 0$

Theorem: $f \in \mathcal{C}^1 \Rightarrow f \in \mathcal{D}$

Example: Let $f(x, y) = g\left(\sqrt{x^2 + y^2}\right), g(x) = x\sin\frac{1}{x}, g(0) = 0$. Show that $f \in \mathcal{D}$ but $f \notin \mathcal{C}^1$

Definition: Higher order derivatives

$f_{11} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), f_{12} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), f_{21} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), f_{22} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$ and so on.

We write $f \in \mathcal{C}^2$ to mean $f_{11}, f_{12}, f_{21}, f_{12} \in \mathcal{C}$.

Note:

1. In a similar manner we write $f \in \mathcal{C}^n$ to mean that all the(2^n of them) n th order partial derivatives are continuous.
2. That there are $\binom{n}{m} = {}^n C_m = \frac{n!}{m!(n-m)!}$,n th order partial derivatives containing x, m times.

Example: Let $f(x,y) = \begin{cases} 2xy\frac{x^2-y^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

Show that $f_{12}(0,0) \neq f_{21}(0,0)$.

Theorem: $f \in \mathcal{C}^2 \Rightarrow f_{12} = f_{21}$

Theorem: Chain Rule

1. $f = f(x,y), x = x(t), y = y(t)$ all in $\mathcal{C}^1$
   $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
2. $f = f(x,y), x = x(u,v), y = y(u,v)$ all in $\mathcal{C}^1$
   $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$ and $\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$

Note: The above may be written as

1. $\frac{\partial f}{\partial t} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial t}$
2. $\frac{\partial f}{\partial(u,v)} = \begin{pmatrix} \frac{\partial f}{\partial u} \\ \frac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial f}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}$

With $\underline{x} = \binom{x}{y}$ and $\underline{u} = \binom{u}{v}$ they may also be written as

1. $(f \circ \underline{x})'(t) = f'(\underline{x})\underline{x}'(t)$
2. $(f \circ \underline{x})'(\underline{u}) = f'(\underline{x})\underline{x}'(\underline{u})$

We see that $\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \frac{\partial f}{\partial(x,y)} = f'(\underline{x})$ is acting as the true single first derivative of $f = f(x,y)$

Definition: Gradient

$\text{grad} f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix}$

Definition: Directional derivative of f in the direction of the unit vector $\underline{u} = (u,v)$ at (a,b)

$D_{\underline{u}} f(a,b) = \lim_{\Delta t \to 0} \frac{f(a+u\Delta t, b+v\Delta t) - f(a,b)}{\Delta t}$

Theorem: $f \in \mathcal{C}^1$

1. $D_{\underline{u}} f(a,b) = \frac{\partial f}{\partial x}(a,b)u + \frac{\partial f}{\partial y}(a,b)v = \nabla f(a,b) \cdot \underline{u}$
2. $\max_{\underline{u}} D_{\underline{u}} f(a,b) = D_{\widehat{\nabla f(a,b)}} f(a,b) = \|\nabla f(a,b)\|$
3. $\min_{\underline{u}} D_{\underline{u}} f(a,b) = D_{-\widehat{\nabla f(a,b)}} f(a,b) = -\|\nabla f(a,b)\|$

Theorem: Normal vector to a surface

$\underline{r} = \underline{r}(t) = \big(x(t), y(t), z(t)\big)$ is a curve on the surface of $z = f = f(x,y) \in \mathcal{C}^1$
where $\underline{r}(t_0) = \big(x(t_0), y(t_0), z(t_0)\big) = \big(a, b, f(a,b)\big)$ then
$\underline{n}(a,b) \cdot \underline{r}'(t_0) = \big(f_x(a,b), f_y(a,b), -1\big) \cdot (x'(t_0), y'(t_0), z'(t_0)) = z'(t_0) - z'(t_0) = 0$
ie $\underline{n}(a,b) = \big(f_x(a,b), f_y(a,b), -1\big) = (\nabla f(a,b), -1)$ is a vector perpendicular to the surface $f = f(x,y)$ at (a,b)

Theorem: Equation of the tangent plane to the surface of $z = f = f(x,y)$ at (a,b)

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = f(a,b) + \nabla f(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix}$

Theorem: Taylor's expansion for one variable $f: I \in \mathbb{R} \to \mathbb{R}$
$f \in C^{n+1}$ and $a, a+h \in I$
then $f(a+h) = \sum_{i \le n} \frac{1}{i!}\frac{d^i f}{dx^i}(a)h^i + \frac{1}{(n+1)!}\frac{d^{n+1}f}{dx^{n+1}}(a+\theta h)h^{n+1}$
where $\theta \in (0,1)$

Note: $\sum_{i \le n} \frac{1}{i!}\frac{d^i f}{dx^i}(a)h^i = \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(a)h^i$

Note: The first two terms are the equation of the tangent line.

Proof: Use generalized mean value theorem on
$F(t) = \sum_{i=0}^{n} \frac{1}{i!} f^{(i)}(t)(x-t)^i$ and $G(t) = (x-t)^{n+1}$

Example: When $n = 1$
$f(a+h) = \sum_{i \le 1} \frac{1}{i!}\frac{d^i f}{dx^i}(a)h^i + \frac{1}{2!}\frac{d^2 f}{dx^2}(a+\theta h)h^{n+1} = f(a) + \frac{1}{1!}f'(a)h + \frac{1}{2!}f''(c)h^2$

Example: Write the Taylor's expansion for $f(x) = e^x$ at $a = 0$.

Example: Derive the second derivative test to find the extrema of $f(x)$. What to do when $f''(a) = 0$?

Theorem: Taylor's for two variables $f: A \in \mathbb{R}^2 \to \mathbb{R}$
$f \in C^{n+1}$ and $(a,b), (a+h, b+k) \in A$
then $f(a+h, b+k) = \sum_{i+j \le n} \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a,b)h^i k^j + \sum_{i+j=n+1} \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a+\theta h, b+\theta k)h^i k^j$
where $\theta \in (0,1)$

Note: $\sum_{i+j \le n} \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a,b)h^i k^j = \sum_{r=0}^{n} \frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}\frac{\partial^r f}{\partial x^{r-j}\partial y^j}(a,b)h^{r-j}k^j = \sum_{r=0}^{n}\frac{1}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial x}\right)^r f(a,b)$

Proof: Use Taylor'r expansion for $F(t) = f(a+th, b+tk)$

Example: When $n = 1$
$f(a+h, b+k)$
$= \sum_{i+j \le 1} \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a,b)h^i k^j + \sum_{i+j=2} \frac{1}{i!j!}\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(a+\theta h, b+\theta k)h^i k^j$
$= f(a,b) + \frac{\partial^1 f}{\partial x^1}(a,b)h^1 + \frac{\partial^1 f}{\partial y^1}(a,b)k^1 + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(\mathbf{c})h^2 + \frac{1}{1!1!}\frac{\partial^2 f}{\partial x^1 \partial y^1}(\mathbf{c})hk + \frac{1}{2!}\frac{\partial^2 f}{\partial y^2}(\mathbf{c})k^2$
$= f(a,b) + f_x(a,b)h + f_y(a,b)h + \frac{1}{2!}\left(f_{xx}(\mathbf{c})h^2 + 2f_{xy}(\mathbf{c})hk + f_{yy}(\mathbf{c})k^2\right)$
$= f(a,b) + (f_x(a,b) \quad f_y(a,b))\begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}(h \quad k)\begin{pmatrix} f_{xx}(\mathbf{c}) & f_{xy}(\mathbf{c}) \\ f_{yx}(\mathbf{c}) & f_{yy}(\mathbf{c}) \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$
$= f(a,b) + \nabla f(a,b)\begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}(h \quad k)Hf(\mathbf{c})\begin{pmatrix} h \\ k \end{pmatrix}$
$= f(a,b) + \frac{1}{1!}f'(a,b)\begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2!}(h \quad k)f''(\mathbf{c})\begin{pmatrix} h \\ k \end{pmatrix}$

Note: The first two terms are the equation of the tangent plane.

Definition: $f'' = Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$: Hessian of f
$\det Hf = f_{xx}f_{yy} - {f_{xy}}^2$: determinant
$\mathrm{tr} Hf = f_{xx} + f_{yy}$: trace

Example: Write the Taylor's expansion for $f(x,y) = e^{xy}$ and $f(x,y) = \sin xy$ at $(a,b) = (0,0)$

Definition: (a,b) is a critical point of $f \Leftrightarrow \nabla f(a,b) = \mathbf{0}$ or f is not defined

Definition:

1. f has a relative maximum at $(a,b) \Leftrightarrow f(a,b) \geq f(a+h,b+k)$ in a neighbourhood of (a,b)
2. f has a relative minimum at $(a,b) \Leftrightarrow f(a,b) \leq f(a+h,b+k)$ in a neighbourhood of (a,b)
3. f has a saddle point at $(a,b) \Leftrightarrow f$ is both above and below the tangent plane of f at (a,b) in a neighbourhood of (a,b).

Theorem: $f \in C^1$and (a,b) is a relative maximum/minimum of $f \Rightarrow \nabla f(a,b) = \mathbf{0}$

Theorem: $f \in C^2$ and $\nabla f(a,b) = \mathbf{0}$ then

1. $\det Hf(a,b) > 0$ and $\mathrm{tr} Hf(a,b) > 0$ then (a,b) is a relative mimimum
2. $\det Hf(a,b) > 0$ and $\mathrm{tr} Hf(a,b) < 0$ then (a,b) is a relative maximum
3. $\det Hf(a,b) < 0$ then (a,b) is a saddle point
4. $\det Hf(a,b) = 0$ inconclusive(why?)

Example: Find the relative maxima/minima/saddle points of
$f(x,y) = x^3 - 12x + y^3 - 27y + 5$
$f(x,y) = x^4 + y^4$

Theorem: Lagrange Multipliers
$f, g \in C^1$ and $g_1^2 + g_2^2 > 0$ in A then
The set of points (x,y) on the curve $g(x,y) = 0$ where $f(x,y)$ has maxima or minima, is included in the set of simultaneous solutions (x,y,λ) of the equations $\nabla f(x,y) = \lambda \nabla g(x,y)$, $g(x,y) = 0$

Example:
Find the shortest distance from the point (1,0) to the parabola $y^2 = 4x$
Find the directions of the axes of the ellipse $5x^2 - 6xy + 5y^2 - 4x - 4y - 4 = 0$

Definition/Theorem: Exact Differential Equation
$P(x,y)dx + Q(x,y)dy = 0$ with $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
Solution $f = c$ where $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial x} = Q$

Example:
$(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$
$ydx + 2xdy = 0$

Theorem:
If $P(x,y)dx + Q(x,y)dy = 0$ not exact ie $\frac{\partial P}{\partial x} \neq \frac{\partial Q}{\partial y}$
and
$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/Q = f(x)$ is a function of x alone, let $I = e^{\int f(x)dx}$
or
$\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)/P = g(y)$ is a function of y alone, let $I = e^{\int g(y)dy}$
then $IPdx + IQdy = 0$ is exact ie $\frac{\partial IP}{\partial x} = \frac{\partial IQ}{\partial y}$

Example:
$ydx - (2x + y)dy = 0$
$(x^3 + y^3)dx - xy^2dy = 0$