1. Let $ab =$ last two digits of your index number.

Solve the differential equation $\frac{d^2u}{dt^2} + \frac{du}{dt} + u = 0, u(0) = a, u'(0) = b$

as a system of differential equations $\dot{y} = Ay, y(0) = \left(u(0), u'(0)\right)^T$

Solution:

Assume $ab = 12$

Let $\dot{u} = \frac{du}{dt} = v$ so we have

$\dot{u} = v$,$u(0) = a$ and $\dot{v} = -u - v$,$v(0) = u'(0) = b$.

We can write the system as

$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ with $\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ or

$\dot{y} = Ay$ with $y(0) = \begin{pmatrix} a \\ b \end{pmatrix}$ where $y = \begin{pmatrix} u \\ v \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

Let $y = xe^{wt}$ then we have $\dot{y} = xwe^{wt} = Ay = Axe^{wt}$ or $Ax = wx = \lambda x$

So $w = \lambda$ are the eigenvalues of A and x are the corresponding eigenvectors.

We have det $(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda(1+\lambda) + 1 = \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm i\sqrt{3}}{2}$

For $\lambda_1 = \frac{-1+i\sqrt{3}}{2}$ we have

$(A - \lambda I)x = \begin{pmatrix} -\lambda_1 & 1 \\ -1 & -1-\lambda_1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \overset{-\lambda_1 R_2 + R_1 \to R_2}{\Rightarrow} \begin{pmatrix} -\lambda_1 & 1 \\ 0 & \lambda_1^2 + \lambda_1 + 1 = 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$-\lambda_1 p + q = 0 \Rightarrow x = \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p \\ \lambda_1 p \end{pmatrix} = p \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ so let $x_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$

In the same way for $\lambda_2 = \frac{-1-i\sqrt{3}}{2}$ we have $x_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$

Now since the differential equation is linear, the general solution will be

$y = y(t) = a_1 x_1 e^{\lambda_1 t} + a_2 x_2 e^{\lambda_2 t}$ where a_1, a_2 are constants to be determined.

We can also write the solution as

$$y(t) = (x_1 \quad x_2) \begin{pmatrix} a_1 e^{\lambda_1 t} \\ a_2 x_2 e^{\lambda_2 t} \end{pmatrix} = (x_1 \quad x_2) \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = P \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$y(0) = P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow P \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = P^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{-i\sqrt{3}} \begin{pmatrix} \lambda_2 - 2 \\ -\lambda_1 + 2 \end{pmatrix} = \frac{1}{-i\sqrt{3}} \begin{pmatrix} \frac{-5-i\sqrt{3}}{2} \\ \frac{5-i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{-5i+\sqrt{3}}{2\sqrt{3}} \\ \frac{5i+\sqrt{3}}{2\sqrt{3}} \end{pmatrix}$$

So $a_1 = \frac{-5i+\sqrt{3}}{2\sqrt{3}}$ and $a_2 = \frac{5i+\sqrt{3}}{2\sqrt{3}}$

Note: Check whether you get a real solutions

2. Let $x =$ (last digit of your index number)Mod $3 + 1$. Select matrix number x and call it A.

$$\begin{pmatrix} -11 & -10 & 5 \\ 5 & 4 & -5 \\ -20 & -20 & 4 \end{pmatrix}, \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 & -1 \\ 2 & 5 & -2 \\ 1 & 1 & 2 \end{pmatrix}$$

Solve the system of differential equations $\ddot{y} = Ay, y(0) = (1,2,3)^T, y'(0) = (4,5,6)^T$.

Solution:

Assume $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$ so we have the eigenvalues and the corresponding eigenvectors

$$\lambda_1 = -2, x_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \lambda_2 = -2, x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \lambda_3 = -2, x_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

With $y = xe^{wt}$ we have $\dot{y} = xwe^{wt}$ and $\ddot{y} = xw^2e^{wt}$or $\ddot{y} = xw^2e^{wt} = Ay = Axe^{wt}$ or $Ax = w^2x = \lambda x$

So $w^2 = \lambda (ie\ w = \pm\sqrt{\lambda})$ are the eigenvalues of A and x are the corresponding eigenvectors.

Now since the differential equation is linear, the general solution will be

$y = y(t) = a_1x_1e^{\sqrt{\lambda_1}t} + b_1x_1e^{-\sqrt{\lambda_1}t} + a_2x_2e^{\sqrt{\lambda_2}t} + b_2x_2e^{-\sqrt{\lambda_2}t} + a_3x_3e^{\sqrt{\lambda_3}t} + b_3x_3e^{-\sqrt{\lambda_3}t}$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants to be determined. We can also write the solution as

$$y = y(t) = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_1x_1e^{\sqrt{\lambda_1}t} + b_1x_1e^{-\sqrt{\lambda_1}t} \\ a_2x_2e^{\sqrt{\lambda_2}t} + b_2x_2e^{-\sqrt{\lambda_2}t} \\ a_3x_3e^{\sqrt{\lambda_3}t} + b_3x_3e^{-\sqrt{\lambda_3}t} \end{pmatrix} = P\begin{pmatrix} e^{\sqrt{\lambda_1}t} e^{-\sqrt{\lambda_1}t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\sqrt{\lambda_2}t} e^{-\sqrt{\lambda_2}t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\sqrt{\lambda_3}t} e^{-\sqrt{\lambda_3}t} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix}$$

Therefore

$$y(0) = P\begin{pmatrix} 110000 \\ 001100 \\ 000011 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 110000 \\ 001100 \\ 000011 \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

And

$$y'(t) = P\begin{pmatrix} \sqrt{\lambda_1}e^{\sqrt{\lambda_1}t} - \sqrt{\lambda_1}e^{-\sqrt{\lambda_1}t} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_2}e^{\sqrt{\lambda_2}t} - \sqrt{\lambda_2}e^{-\sqrt{\lambda_2}t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_3}e^{\sqrt{\lambda_3}t} - \sqrt{\lambda_3}e^{-\sqrt{\lambda_3}t} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix}$$

Therefore

$$y'(0) = P\begin{pmatrix} \sqrt{\lambda_1} - \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_2} - \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_3} - \sqrt{\lambda_3} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \sqrt{\lambda_1} - \sqrt{\lambda_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_2} - \sqrt{\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_3} - \sqrt{\lambda_3} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{pmatrix} = P^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Finally we have

$$\begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_1} & -\sqrt{\lambda_1} \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_1} & -\sqrt{\lambda_1} \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_2} & -\sqrt{\lambda_2} \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ d_3 \end{pmatrix} \Rightarrow \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_2} & -\sqrt{\lambda_2} \end{pmatrix}^{-1}\begin{pmatrix} c_2 \\ d_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_3} & -\sqrt{\lambda_3} \end{pmatrix}\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_3 \\ d_3 \end{pmatrix} \Rightarrow \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \sqrt{\lambda_3} & -\sqrt{\lambda_3} \end{pmatrix}^{-1}\begin{pmatrix} c_3 \\ d_3 \end{pmatrix}$$

Note: What is the solution if A is not diagonalizable?