1. Prove that $\lambda_{min}\|x\|^2 \leq x^T Ax \leq \lambda_{max}\|x\|^2$ if $A^T = A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n$

Solution:

Every real symmetric matrix is orthogonally diagonalizable(Theorem)
Ie. there is a orthogonal matrix $Q(Q^{-1} = Q^T)$ consisting of eigenvactors(which are orthonormal) and a diagonal matrix Λ consisting of eigenvalues(real but not necessary distinct) such that $A = Q\Lambda Q^T$

Therefore

$$
\begin{aligned}
& x^T Ax \\
&= x^T Q\Lambda Q^T x \\
&= (Q^T x)^T \Lambda (Q^T x) \\
&= y^T \Lambda y \\
&= \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \\
&= \begin{pmatrix} y_1 & \cdots & y_n \end{pmatrix} \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n \end{pmatrix} \\
&= \sum_{i=1}^{n} \lambda_i y_i^2 \\
&\leq \lambda_{max} \textstyle\sum_{i=1}^{n} y_i^2 \\
&= \lambda_{max} y^T y \\
&= \lambda_{max} (Q^T x)^T Q^T x \\
&= \lambda_{max} x^T Q Q^T x \\
&= \lambda_{max} x^T I x \\
&= \lambda_{max} x^T x \\
&= \lambda_{max} \|x\|^2
\end{aligned}
$$

Left hand side can be proved in a similar manner.

Note:

1. When $x = x_{max}$ the eigenvale corresponding to λ_{max} we have
$x^T Ax = x_{max}^T A x_{max} = x_{max}^T \lambda_{max} x_{max} = \lambda_{max}\|x_{max}\|^2$
So we get the equality in the above case. Same on the left hand side.
2. This means that for a real quadratic form to be positive definite: $0 < x^T Ax$ (or $A \succ 0$) we need $0 < \lambda_{min}\|x\|^2$ or $0 < \lambda_{min}$ or $0 < \lambda$ for all eigenvalues.
Similar results hold for other "definite" cases.
3. Columns of Q are orthonormal by construction. Rows of Q are also orthonormal(why?).

2. Identify the surface $f(x,y,z) = 2x^2 + 12xy + y^2 - 4xz - 8yz - 3z^2 = 0$ by rotating the coordinate axis.

Solution:

We can write the above function as a real quadratic from using a real symmetric matrix

$$f(x,y,z) = 2x^2 + 12xy + y^2 - 4xz - 8yz - 3z^2$$
$$= (x \quad y \quad z)\begin{pmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X^T AX$$

Where $A = \begin{pmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{pmatrix}$ which has the eigenvalue matrix $\Lambda = \begin{pmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and the corresponding eigenvector matrix $P = \begin{pmatrix} -2 & -1 & 2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{pmatrix}$. The columns of P are orthogonal. We can make the columns of P orhonormal by dividing each column by its magnitude to make a orthogonal matrix $Q = \begin{pmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix}$. This is the orthogonal diagonalization $AQ = Q\Lambda$ or $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ we require

By the discussion on Q1, with $X' = Q^T X$, we can write

$$f(x,y,z) = X^T AX = X'^T \Lambda X' = (x' \quad y' \quad z')\begin{pmatrix} 9 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 9x'^2 - 6y'^2 - 3z'^2 = 0$$

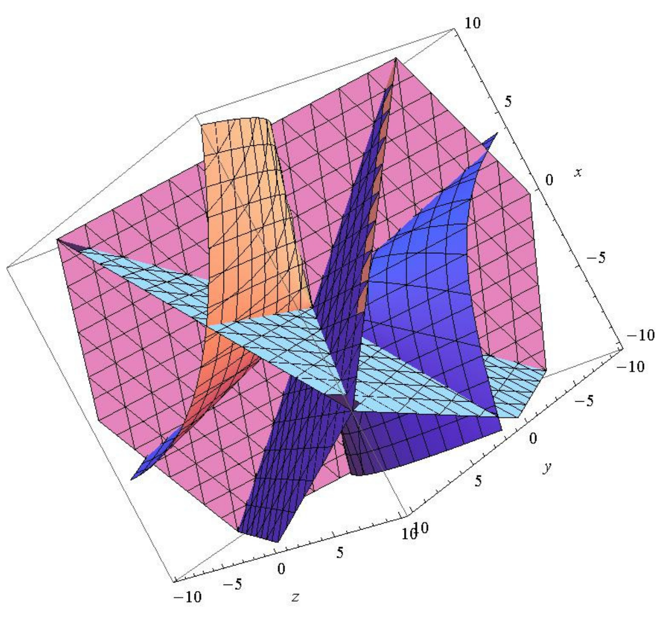
or $9x'^2 = 6y'^2 + 3z'^2$. This is a cone ($9x'^2 = r^2$) along the x' axix and ellipses($6y'^2 + 3z'^2 = r^2$) on the $y'z'$ plane making $f(x,y,z) = 0$ an elliptical cone.

Note:

Transformation from the xyz coordinates to $x'y'z'$ coordinates is given by $X' = Q^T X$ or

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 2/3 \end{pmatrix}^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2/3 & -2/3 & 1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z \\ -\frac{1}{3}x + \frac{2}{3}y + \frac{2}{3}z \\ \frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}z \end{pmatrix}$$

The x' axis is perpendicular to the plane $-\frac{2}{3}x - \frac{2}{3}y + \frac{1}{3}z = 0$ and so on. In other words x', y', z' axis will be along the intersection of the 3 planes.



Note:

We have that the eigenvalues of A are $9, -6, -3$. Therefore from the discussion on Note from Q1, we see that the quadratic form $f(x,y,z) = X^T AX$ is indefinite, which means that it will attain both positive, negative values in 4D. The graph above is the image when it is 0.