1. Let $T:\mathbb{R}^2 \to \mathbb{R}^3;\ T:\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-2y \\ 2x+3y \end{pmatrix}$ be a linear Transfromation.

   Find the matrix of T with respect to the bases $B'_{\mathbb{R}^2} = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$ and $B'_{\mathbb{R}^3} = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}\right\}$

Solution

$$(T(u_i'))_2 = \left(T\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad T\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 & 2 \\ -3 & -1 \\ 8 & 5 \end{pmatrix} = (w_j')_3 A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{pmatrix} A'$$

$$A' = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ -3 & -1 \\ 8 & 5 \end{pmatrix} = \frac{1}{(-6)1-(3).1+(2).0} \begin{pmatrix} +(-6) & -(3) & +(2) \\ -(3) & +(3) & -(2) \\ +(3) & -(3) & +(-1) \end{pmatrix}^T \begin{pmatrix} 3 & 2 \\ -3 & -1 \\ 8 & 5 \end{pmatrix}$$
$$= \frac{1}{-9}\begin{pmatrix} -6 & -3 & 3 \\ -3 & 3 & -3 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -3 & -1 \\ 8 & 5 \end{pmatrix} = \frac{1}{-9}\begin{pmatrix} 15 & 6 \\ -42 & -24 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -5/3 & -2/3 \\ 14/3 & 8/3 \\ -4/9 & -1/9 \end{pmatrix}$$

Note: Theorem on Change of Basis

$$(T(u_i))_2 = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 3 \end{pmatrix} = (w_j)_3 A = IA = A$$

$$(w_j')_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{pmatrix} = (w_j)_3 Q = IQ = Q$$

$$(u_i')_2 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = (u_i)_2 P = IP = P$$

$(T(u_i'))_2 = (w_j')_3 A'$
$(T(u_i))_2 P = (w_j)_3 QA'$
$(w_j)_3 AP = (w_j)_3 QA'$
$(w_j)_3 (AP - QA') = \underline{0}$
$AP - QA' = O$ or $AP = QA'$ or $A' = Q^{-1}AP$ since $(w_j)_3$ is a basis

So $A' = Q^{-1}AP = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ -3 & -1 \\ 8 & 5 \end{pmatrix}$

Note:

$$(T(u_i'))_2 = (w_j')_3 A' = (w_j')_3 \begin{pmatrix} -5/3 & -2/3 \\ 14/3 & 8/3 \\ -4/9 & -1/9 \end{pmatrix}$$

Therefore

$T(u_1') = -\frac{5}{3}w_1' + \frac{14}{3}w_2' - \frac{4}{9}w_3'$
$T(u_2') = -\frac{2}{3}w_1' + \frac{8}{3}w_2' - \frac{1}{9}w_3'$

2. Let $\mathrm{Hom}(U,V)$ be the set of all linear transformations (Homomorphisms) from U to V over the field F. For $S,T \in \mathrm{Hom}(U,V)$ and $a \in F$ let the addition and scalar multiplication $S+T$ and aT be fined as $(S+T)(x) = S(x) + T(x)$ and $(aT)(x) = aT(x)$ for $\forall x \in V$ and $\forall a \in F$. Show that $\mathrm{Hom}(U,V)$ is a vector space over F with the above defined addition and scalar multiplication.
Note: A linear transformation $\mathrm{Hom}(U,F)$ is called the dual space of U or U^* and the elements of that are called Linear Operators.

Solution

1. $(\mathrm{Hom}(U,V),+)$ is an Abelian Group:

1.1 $\exists O \in \mathrm{Hom}(U,V); O: U \to V, O(x) = \underline{0} \Rightarrow \mathrm{Hom}(U,V) \neq \emptyset$

1.2 $\forall S,T \in \mathrm{Hom}(U,V); S+T \in \mathrm{Hom}(U,V)$

1.3 $\forall R,S,T \in \mathrm{Hom}(U,V); \forall x \in U$:
$\big(R+(S+T)\big)(x) = R(x) + (S+T)(x) = R(x) + \big(S(x)+T(x)\big) = (R(x)+S(x)) + T(x)$
$= (R+S)(x) + T(x) = \big((R+S)+T\big)(x)$
$\Rightarrow R+(S+T) = (R+S)+T$

1.4 $\exists O \in \mathrm{Hom}(U,V); O: U \to V, O(x) = \underline{0};\ \forall S \in \mathrm{Hom}(U,V); \forall x \in U$:
$(S+O)(x) = S(x) + O(x) = S(x) + \underline{0} = S(x)$
$\Rightarrow S + O = S$

1.5 $\forall S \in \mathrm{Hom}(U,V); \exists -S \in \mathrm{Hom}(U,V); -S: U \to V, (-S)(x) = -S(x)$:
$(-S+S)(x) = (-S)(x) + S(x) = -S(x) + S(x) = \underline{0} = O(x)$
$\Rightarrow -S + S = O$

1.6 $\forall S,T \in \mathrm{Hom}(U,V); \forall x \in U$:
$(S+T)(x) = S(x) + T(x) = T(x) + S(x) = (T+S)(x)$
$\Rightarrow S+T = T+S$

2. $(F,+,\cdot)$ is a Field

3. $\forall a \in F; \forall S \in \mathrm{Hom}(U,V); aS \in \mathrm{Hom}(U,V)$

4. $\forall a \in F; \forall S,T \in \mathrm{Hom}(U,V);\ \forall x \in U$:
$\big(a(S+T)\big)(x) = a((S+T)(x)) = a(S(x)+T(x)) = aS(x) + aT(x) = (aS)(x) + (aT)(x) =$
$(aS+aT)(x)$
$\Rightarrow a(S+T) = aS + aT$

5. $\forall a,b \in F; \forall S \in \mathrm{Hom}(U,V); \forall x \in U$:
$\big((a+b)S\big)(x) = (a+b)S(x) = aS(x) + bS(x) = (aS)(x) + (bS)(x) = (aS+bS)(x)$
$\Rightarrow (a+b)S = aS + bS$

6. $\forall a,b \in F; \forall S \in \mathrm{Hom}(U,V); \forall x \in U$:
$\big((ab)S\big)(x) = (ab)S(x) = a\big(bS(x)\big) = a\big((bS)(x)\big) = a(bS)(x) = \big(a(bS)\big)(x)$
$\Rightarrow (ab)S = a(bS)$

7. $\forall S \in \mathrm{Hom}(U,V);\ \forall x \in U$:
$(1S)x = 1S(x) = S(x)$
$\Rightarrow 1S = S$

$\therefore$ $\mathrm{Hom}(U,V)$ is a vector space over F

Note: We don't show the commutativity in 1.4 and 1.5 since we show it in 1.6 in general.