MODEL QUESIONS & ANSWERS MA(101)

(MATRICES)

1. Given $A=\begin{pmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{pmatrix}$ $B=\begin{pmatrix}3 & -1 & 2\\ 4 & 2 & 5\\ 2 & 0 & 3\end{pmatrix}$ and $C=\begin{pmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{pmatrix}$

(a) Compute that A+B and A-C

(b) Verify that A+(B+C) = (A+B)+C

(c) Compute AB, BA and AC^T

Solution: $A+B=\begin{pmatrix}4 & 1 & -1\\ 9 & 2 & 7\\ 3 & 1 & 7\end{pmatrix}$ and $A-C=\begin{pmatrix}-3 & 1 & -5\\ 5 & -3 & 0\\ 0 & 1 & -2\end{pmatrix}$

$$(A+B)+C=A+(B+C)=\begin{pmatrix}8 & 2 & 1\\ 9 & 5 & 9\\ 4 & -3 & 7\end{pmatrix}$$

$$AB=\begin{pmatrix}5 & 3 & 3\\ 19 & -5 & 16\\ 1 & -3 & 0\end{pmatrix}\quad BA=\begin{pmatrix}0 & 4 & -9\\ 19 & 3 & -3\\ 5 & 1 & -3\end{pmatrix}$$

$$AC^T=\begin{pmatrix}0 & 0 & 12\\ 24 & 4 & 11\\ 5 & -1 & 6\end{pmatrix}$$

Q(2). Evaluate :

$$\begin{pmatrix}0 & 1 & 3\\ 4 & 5 & 6\\ 7 & 3 & 2\end{pmatrix}\begin{pmatrix}1 & 4 & 5\\ 6 & 3 & 2\\ 2 & 3 & 1\end{pmatrix}=\begin{pmatrix}12 & 12 & 5\\ 46 & 49 & 36\\ 29 & 43 & 43\end{pmatrix}$$

Q(3). If $A = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$, show that $A^2 = \begin{pmatrix} 9 & -4 \\ -8 & 17 \end{pmatrix}$ and $A^3 = \begin{pmatrix} -7 & 30 \\ 60 & -67 \end{pmatrix}$

Q(4). If $A = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 2 \\ -1 & 0 & -1 \\ -3 & -3 & -2 \end{pmatrix}$

Show that $\{kA + (1-k)B\}^2 = I$, k being a scalar.

Solution: $\{kA + (1-k)B\}^2 = k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA$

Now compute each matrix as follows;

$k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA =$

$$A^2 = \begin{pmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -4 & -3 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; AB = \begin{pmatrix} 5 & 4 & 4 \\ -2 & -1 & -2 \\ 4 & 2 & 3 \end{pmatrix}, BA = \begin{pmatrix} 1 & -4 & -4 \\ 0 & 3 & 2 \\ -2 & 2 & 3 \end{pmatrix}$$

Consider element a_{11}; of $k^2A^2 + (1-k)^2B^2 + k(1-k)AB + k(1-k)BA$

$-3k^2 + (1-k)^2 + 5k(1-k) + k(1-k) = 1 - 8k^2 + 4k \equiv 1$

Therefore k =1/2

When this value $\left(A^2 + B^2 + AB + BA\right)\frac{1}{4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Q(5). Find x, y such that $\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$$\begin{aligned} 2x - y &= 8 \\ -3x + 4y &= 1 \end{aligned} \quad x = \frac{31}{5}, \; y = \frac{22}{5}$$

Q(6). Express the matrix $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ -3 & 1 & 4 \end{pmatrix}$ as the sum of symmetric and skew symmetric matrix.

$$S = \begin{pmatrix} 1 & 2 & 1/2 \\ 2 & 3 & 0 \\ 1/2 & 0 & 4 \end{pmatrix} \text{ and } T = \begin{pmatrix} & 0 & 7/2 \\ 0 & 0 & -1 \\ -7/2 & 1 & 0 \end{pmatrix}$$

Exercises 3.5

Q(1). If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, show that $A^3 = A^{-1}$.

If $A^3 = A^{-1}$ is satisfied then $AA^3 = AA^{-1}$ or $A^4 = I$

Try

$$A^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ therefore, } A^3 = A^{-1}$$

Q(2). Find the Inverse of (a) $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 5 & 3 \\ 1 & 2 & 1 \end{pmatrix}$,

Use direct method $A^{-1} \begin{pmatrix} 1/3 & -2/3 & 4/3 \\ -1/3 & -1/3 & 5/3 \\ 1/3 & 4/3 & -11/3 \end{pmatrix}$

(b) $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

Use elementary row operations

Define (A|I) = $\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 4 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$

$\begin{array}{l} R_2 \to -4R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{array}$ $\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & -10 & -25 & -4 & 1 & 0 \\ 0 & -1 & -6 & -1 & 0 & 1 \end{array}\right)$

$R_3 \leftrightarrow -R_3$ $\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & -10 & -25 & -4 & 1 & 0 \\ 0 & 1 & 6 & 1 & 0 & -1 \end{array}\right)$

$R_3 \longleftrightarrow R_2$ $\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & -1 \\ 0 & -10 & -25 & -4 & 1 & 0 \end{array}\right)$

$\begin{array}{l} R_1 \to 3R_2 + R_1 \\ R_3 \to 10R_1 + R_3 \end{array}$ $\left(\begin{array}{ccc|ccc} 1 & 0 & -11 & -2 & 0 & 3 \\ 0 & 1 & 6 & 1 & 0 & -1 \\ 0 & 0 & 1 & 6/35 & 1/35 & -10/35 \end{array}\right)$

$\begin{array}{l} R_1 \to 11R_3 + R_1 \\ R_2 \to -6R_3 + R_2 \end{array}$ $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/35 & 11/35 & -5/35 \\ 0 & 1 & 0 & -1/35 & -6/35 & 25/35 \\ 0 & 0 & 1 & 6/35 & 1/35 & -10/35 \end{array}\right)$

Therefore, $A^{-1} = \begin{pmatrix} -4/35 & 11/35 & -5/35 \\ -1/35 & -6/35 & 25/35 \\ 6/35 & 1/35 & -10/35 \end{pmatrix}$

Find the inverse of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ by elementary row transformations.

Let $A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

Define $(A|I) = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$

$R_2 \to -2R_1 + R_2 \quad \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$

$R_1 \leftrightarrow -R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$

$R_2 \to -2R_1 + R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & -2 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right)$

$R_3 \leftrightarrow -R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & -2 & -5 & 5 & -2 & 0 \end{array}\right)$

$R_3 \to 2R_2 + R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right)$

$$R_1 \to -R_2 + R_1 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & -1 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right)$$

$$\begin{array}{l} R_2 \to -3R_3 + R_2 \\ R_1 \to R_2 + R_1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array}\right)$$

Therefore, $A^{-1} = \underline{\underline{\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}}}$

Q(5). If $A = \begin{pmatrix} 1 & -a & 1 \\ b & 0 & 2b \\ 0 & a & 0 \end{pmatrix}$ then, show that

(i) $A^3 = abA + A^2 - abI$, is satisfied

Use method of induction .Multiply the above equation by A

$A^4 = abA^2 + A^3 - abA$ and eliminate A^3

It gives, $A^4 - abA^2 - A^2 + abI = 0$

Therefore, the given statement is true hen n = 4.

Assume that it is also true upto n =p (even) number

$A^p - abA^{p-2} - A^2 + abI = 0$

Multiply this relation with A^2, we get, $A^{p+2} - abA^p - A^4 + abA^2 = 0$

Now eliminate A^4

$A^{p+2} - abA^p - A^2 + abI = 0$

Therefore it I true for n =p+2.

Hence by Methodical induction the given expression is true for every integer values of n

(ii) Show also that $A^{2n} = \left\{\frac{(ab)^n - 1}{ab-1}\right\}A^2 - \left\{\frac{ab\{(ab)^{n-1} - 1\}}{ab-1}\right\}I$, where n is a positive integer.

When n = 2, $A^4 = \left\{\frac{(ab)^2 - 1}{ab-1}\right\}A^2 - \left\{\frac{ab\{(ab) \ \ - 1\}}{ab-1}\right\}I$

$A^4 = (ab+1)A^2 - abI$ which satisfied the above expression.

Assume that the statement is true upto n = p

$$A^{2p} = \left\{\frac{(ab)^p - 1}{ab-1}\right\}A^2 - \left\{\frac{ab\{(ab)^{p-1} - 1\}}{ab-1}\right\}I \qquad \text{(A)}$$

Multiply (A) with A^2

$$A^{2(p+1)} = \left\{\frac{(ab)^p - 1}{ab-1}\right\}A^4 - \left\{\frac{ab\{(ab)^{p-1} - 1\}}{ab-1}\right\}A^2$$

Eliminate A^4 and get,

$$A^{2(p+1)} = \left\{\frac{(ab)^p - 1}{ab-1}\right\}\left((ab+1)A^2 - abI\right) - \left\{\frac{ab\{(ab)^{p-1} - 1\}}{ab-1}\right\}A^2$$

$$A^{2(p+1)} = \left\{\frac{((ab)^p - 1)(ab+1) - ab\{(ab)^{p-1} - 1\}}{ab-1}\right\}A^2 - \left\{\frac{ab\{(ab)^p - 1\}}{ab-1}\right\}I$$

$$A^{2(p+1)} = \left\{\frac{(ab)^{p+1} - 1}{ab-1}\right\}A^2 - \left\{\frac{ab\{(ab)^p - 1\}}{ab-1}\right\}I$$

Therefore, the above statement is true for n = p+1, hence by method if induction it is true for every positive integers n, i.e, $A^{2n} = \left\{\dfrac{(ab)^n - 1}{ab-1}\right\}A^2 - \left\{\dfrac{ab\{(ab)^{n-1} - 1\}}{ab-1}\right\}I$

6.(a) Show that every 2x2 matrix such that $X^T AX = B,$ where

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has one of the forms $\begin{pmatrix} a & \dfrac{1}{2a} \\ a & \dfrac{1}{-2a} \end{pmatrix}$ or $\begin{pmatrix} a & \dfrac{1}{2a} \\ -a & \dfrac{1}{2a} \end{pmatrix}$

Solution:

Let form of matrix X is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then

$X^T AX = B, \Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^2 - c^2 & ab - cd \\ ab - cd & b^2 - d^2 \end{pmatrix}$

Equating respective elements,

$a^2 - c^2 = 0$

$b^2 - d^2 = 0$

$ac - bd = 1$

If a = t then $c^2 = t^2$, and $c = \pm t$

Suppose b =k $b^2 = d^2 \Rightarrow d = \pm k$ and tk-(-tk)=2tk=1

$\Rightarrow t = \dfrac{1}{2k}$

Therefore form of matrix is $\begin{pmatrix} t & \dfrac{1}{2t} \\ t & \dfrac{1}{-2t} \end{pmatrix}$ or $\begin{pmatrix} t & \dfrac{1}{2t} \\ -t & \dfrac{1}{2t} \end{pmatrix}$

(b) If $P = QRQ^{-1}$, show that $P^n = QR^nQ^{-1}$ where n is a positive integer.

Try $P^2 = (QRQ^{-1})(QRQ^{-1}) = QR^2Q^{-1}$

Similarly $P^3 = (QR^2Q^{-1})(QRQ^{-1}) = QR^3Q^{-1}$

In general $\underline{\underline{P^n = QR^nQ^{-1}}}$

(c) Let $P = \begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix}$, show that $Q^{-1}PQ = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$. Hence

find P^n.

When $Q = \begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix}$, $\Rightarrow$ $Q^{-1} = -\frac{1}{5}\begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix}$

$$Q^{-1}PQ = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ 7 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

Hence $P = QRQ^{-1}$, and $P^n = QR^nQ^{-1}$

$$P^n = QR^nQ^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 7 & 1 \end{pmatrix}\begin{pmatrix} (-2)^n & 0 \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 2(-2)^n - 7\times 3^n & 0(-2)^{n+1} + 2\times 3^n \\ 7(-2)^n - 7\times 3^n & -7(-2)^n + 2\times 3^n \end{pmatrix}$$