MODEL QUESIONS & ANSWERS MA(101)

DETERMINANTS

Q(1). Evaluate (a) $\begin{vmatrix} 28 & 25 & 38 \\ 42 & 38 & 65 \\ 56 & 47 & 83 \end{vmatrix}$ =770.

(b) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = C_2 \to C_2 + C_3 \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$

$\begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} = 0$ Since there are two columns are equal.

Q(2). Prove that without expanding, that each of the following determinants vanishes. (a) $\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix} = \begin{matrix} C_4 \to -C_1 + C_4 \\ C_3 \to -C_2 + C_3 \end{matrix} \begin{vmatrix} 1 & 15 & -1 & 3 \\ 12 & 6 & 1 & -3 \\ 8 & 10 & 1 & -3 \\ 13 & 3 & -1 & 3 \end{vmatrix}$

$\begin{vmatrix} 1 & 15 & -1 & 3 \\ 12 & 6 & 1 & -3 \\ 8 & 10 & 1 & -3 \\ 13 & 3 & -1 & 3 \end{vmatrix} = -3\begin{vmatrix} 1 & 15 & -1 & -1 \\ 12 & 6 & 1 & 1 \\ 8 & 10 & 1 & 1 \\ 13 & 3 & -1 & -1 \end{vmatrix} = 0$ Since there are two columns are equal.

(b) $$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} = \begin{matrix} R_2 \to -R_1+R_2 \\ R_3 \to -R_1+R_3 \\ R_4 \to -R_1+R_4 \end{matrix}$$

$$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b-a & b^2-a^2 & b^3+cda-a^3-bcd \\ 1 & c-a & c^2-a^2 & c^3+dab-a^3-bcd \\ 1 & d-a & d^2-a^2 & d^3+abc-a^3-bcd \end{vmatrix} = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & 1 & b+a & b^2+ab+a^2-cd \\ 1 & 1 & c+a & c^2+ac+a^2-bd \\ 1 & 1 & d+a & d^2+ad+a^2-bc \end{vmatrix}$$

$$C_1 \to -C_2+C_1 = (b-a)(c-a)(d-a)\begin{vmatrix} 1-a & a & a^2 & a^3+bcd \\ 0 & 1 & b+a & b^2+ab+a^2-cd \\ 0 & 1 & c+a & c^2+ac+a^2-bd \\ 0 & 1 & d+a & d^2+ad+a^2-bc \end{vmatrix}$$

Expand through column 1,

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+a^2+ab-cd \\ 1 & c+a & c^2+a^2+ac-bd \\ 1 & d+a & d^2+a^2+ad-cb \end{vmatrix}$$

$R_2 \to -R_1+R_2$
$R_3 \to -R_1+R_3$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & b^2+a^2+ab-cd \\ 0 & c-b & c^2-b^2+ac-ab-bd+cd \\ 0 & d-b & d^2-b^2+ad-ab+cd-cb \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & b+a & b^2+a^2+ab-cd \\ 0 & 1 & c+b+a+d \\ 0 & 1 & d+b+a+c \end{vmatrix} = 0$$ Since row 2 and row 3 are same.

Q(3). Show that $\begin{vmatrix} x^2+1 & xy & xz & xu \\ xy & y^2+1 & yz & yu \\ zx & zy & z^2+1 & zu \\ ux & uy & uz & u^2+1 \end{vmatrix} = x^2+y^2+z^2+u^2+1$.

Multiply each row by x, y, z and u respectively and divide with the same, then

$= \frac{1}{xyzu}\begin{vmatrix} x^3+x & x^2y & x^2z & x^2u \\ xy^2 & y^3+y & y^2z & y^2u \\ z^2x & z^2y & z^3+z & z^2u \\ u^2x & u^2y & u^2z & u^3+u \end{vmatrix}$ Takeout factor x, y, z and u from the columns 1 ,2, 3, and 4.

$$= \frac{1}{xyzu}\begin{vmatrix} x^3+x & x^2y & x^2z & x^2u \\ xy^2 & y^3+y & y^2z & y^2u \\ z^2x & z^2y & z^3+z & z^2u \\ u^2x & u^2y & u^2z & u^3+u \end{vmatrix} = \begin{vmatrix} x^2+1 & x^2 & x^2 & x^2 \\ y^2 & y^2+1 & y^2 & y^2 \\ z^2 & z^2 & z^2+1 & z^2 \\ u^2 & u^2 & u^2 & u^2+1 \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$= \begin{vmatrix} x^2+1 & x^2 & x^2 & x^2 \\ y^2 & y^2+1 & y^2 & y^2 \\ z^2 & z^2 & z^2+1 & z^2 \\ u^2 & u^2 & u^2 & u^2+1 \end{vmatrix} = (1+x^2+y^2+z^2+u^2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y^2 & y^2+1 & y^2 & y^2 \\ z^2 & z^2 & z^2+1 & z^2 \\ u^2 & u^2 & u^2 & u^2+1 \end{vmatrix}$$

$C_4 \to -C_1 + C_4$
$C_3 \to -C_1 + C_3$
$C_2 \to -C_1 + C_2$

$$(1+x^2+y^2+z^2+u^2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ y^2 & y^2+1 & y^2 & y^2 \\ z^2 & z^2 & z^2+1 & z^2 \\ u^2 & u^2 & u^2 & u^2+1 \end{vmatrix} = (\quad)\begin{vmatrix} 1 & 0 & 0 & 0 \\ y^2 & 1 & 0 & 0 \\ z^2 & 0 & 1 & 0 \\ u^2 & 0 & 0 & 1 \end{vmatrix}$$

$$= \underline{\underline{(1+x^2+y^2+z^2+u^2)}}$$

Q(4). Solve the equation

(a) $\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & b+x & a+x \end{vmatrix} = 0,$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} a+b+c+3x & a+b+c+3x & a+b+c+3x \\ b+x & c+x & a+x \\ c+x & b+x & a+x \end{vmatrix} = (a+b+c+3x)\begin{vmatrix} 1 & 1 & 1 \\ b+x & c+x & a+x \\ c+x & b+x & a+x \end{vmatrix}$$

$$= (a+b+c+3x)\begin{vmatrix} 1 & 1 & 1 \\ b+x & c+x & a+x \\ c+x & b+x & a+x \end{vmatrix} = (a+b+c+3x)\begin{vmatrix} 1 & 0 & 0 \\ b+x & c-b & a-b \\ c+x & b-c & a-c \end{vmatrix}$$

$$= (a+b+c+3x)(c-b)\begin{vmatrix} 1 & 0 & 0 \\ b+x & 1 & a-b \\ c+x & -1 & a-c \end{vmatrix}$$

$= (a+b+c+3x)(c-b)(2a-b-c) = 0$

$\Rightarrow x = -(a+b+c)/3$

(b) $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix}$ $R_1 \to -R_2 + R_1$

$$= \begin{vmatrix} -(x+1) & -(x+1) & -(x+1) \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = -(x+1)\begin{vmatrix} 1 & 1 & 1 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix}$$

$C_3 \to -C_1 + C_3$
$C_2 \to -C_1 + C_2$
$= -(x+1)\begin{vmatrix} 1 & 0 & 0 \\ 2x+3 & x+1 & 2x+2 \\ 3x+5 & 2x+3 & 7x+12 \end{vmatrix}$

$$= -(x+1)\begin{vmatrix} 1 & 0 & 0 \\ 2x+3 & x+1 & 2x+2 \\ 3x+5 & 2x+3 & 7x+12 \end{vmatrix}$$

$$= -(x+1)\begin{vmatrix} x+1 & 2(x+1) \\ 2x+3 & 7x+12 \end{vmatrix} = -(x+1)^2\begin{vmatrix} 1 & 2 \\ 2x+3 & 7x+12 \end{vmatrix} = -(x+1)^2(3x+6)$$

$$\underline{\underline{= -2(x+1)^2(x+2)}}$$

Q(5). Prove, without expanding , that each of the following determinants vanishes,

(a) $$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix} \begin{matrix} R_2 \to -R_1+R_2 \\ R_3 \to -R_1+R_3 \end{matrix} = \begin{vmatrix} 1 & a & a^2-bc \\ 0 & b-a & b^2-a^2+bc-ac \\ 0 & c-a & c^2-a^2+bc-ab \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2-bc \\ 0 & 1 & b+a+c \\ 0 & 1 & c+a+b \end{vmatrix} = 0$$ since row 2 and row 3 are same.

(b) $$\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} R_1 \to R_1+R_2+R_3$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$$

Q(6). Prove the followings:

$$\begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix} = \begin{vmatrix} m & b & q \\ l & a & p \\ n & c & r \end{vmatrix} = \begin{vmatrix} l & m & n \\ p & b & r \\ a & b & c \end{vmatrix}$$

Suppose matrix $A = \begin{pmatrix} a & b & c \\ l & m & n \\ p & q & r \end{pmatrix}$ and $A^T = \begin{pmatrix} a & l & p \\ b & m & q \\ c & n & r \end{pmatrix}$ then $|A| = |A^T|$

Hence, $\begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & l & p \\ b & m & q \\ c & n & r \end{vmatrix} C_2 \longleftrightarrow C_1 = -\begin{vmatrix} l & a & p \\ m & b & q \\ n & C & r \end{vmatrix}$

$R_2 \longleftrightarrow R_1 = -\begin{vmatrix} l & a & p \\ m & b & q \\ n & C & r \end{vmatrix} = --\begin{vmatrix} m & b & q \\ b & a & p \\ n & C & r \end{vmatrix} = \begin{vmatrix} m & b & q \\ l & a & p \\ n & C & r \end{vmatrix}$

Similarly by interchanging rows and columns one can prove next also.

Q(7). $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = \begin{vmatrix} a & b+c & c+a \\ b & c+a & a+b \\ c & a+b & b+c \end{vmatrix} + \begin{vmatrix} b & b+c & c+a \\ c & c+a & a+b \\ a & a+b & b+c \end{vmatrix}$

$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix} + \begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$

$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$

Since $\begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} = 0$

$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} + \begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix} + \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$

Since $\begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} = 0$ and $\begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix} = \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix} = 0$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix} + \begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Therefore $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Q(8).

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = a^4 + 4a^3$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$\begin{vmatrix} 4+a & 4+a & 4+a & 4+a \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = (4+a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix}$$

$C_4 \to -C_1 + C_4$
$C_3 \to -C_1 + C_3$
$C_2 \to -C_1 + C_2$

$$(4+a)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+a & 1 \\ 1 & 1 & 1 & 1+a \end{vmatrix} = (4+a)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & 0 & a & 0 \\ 1 & 0 & 0 & a \end{vmatrix}$$

Expansion through first row,

$$= (4+a)\begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = \underline{\underline{a^3(4+a)}}$$

Q(9) . $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$. $\begin{matrix} C_3 \to -C_1 + C_3 \\ C_2 \to -C_1 + C_2 \end{matrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ bc & ca-cb & ab-cb \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ bc & -c & -b \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ bc & -c & -b \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ -c & -b \end{vmatrix}$$

$$\underline{\underline{= (b-a)(c-a)(c-b)}}$$

Q(10). $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$ $R_3 \to R_1 + R_3$

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a+b+c & b+c+a & c+a+b \end{vmatrix} = (a+b+c)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$C_3 \to -C_1 + C_3$
$C_2 \to -C_1 + C_2$

$$= (a+b+c)\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \\ 1 & 0 & 0 \end{vmatrix} = (a+b+c)(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ a^2 & b+a & c+a \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)\begin{vmatrix} a & 1 & 1 \\ a^2 & b+a & c+a \\ 1 & 0 & 0 \end{vmatrix} = (a+b+c)(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (a+b+c)(b-a)(c-a)(c-b)$$

Therefore, $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Q(11). Factorize

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

$R_4 \to -R_1 + R_4$
$R_3 \to -R_1 + R_3$
$R_2 \to -R_1 + R_2$

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} = (b-a)(c-a)(d-a)\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+a^2-ab \\ 0 & 1 & c+a & c^2+a^2-ac \\ 0 & 1 & d+a & d^2+a^2-db \end{vmatrix}$$

Expansion through first column,

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & a^2+b^2-ab \\ 1 & c+a & c^2+a^2-ac \\ 1 & d+a & a^2+d^2-ad \end{vmatrix} \quad \begin{matrix} R_3 \to -R_1 + R_3 \\ R_2 \to -R_1 + R_2 \end{matrix}$$

$$= (b-a)(c-a)(d-a)\begin{vmatrix} 1 & b+a & a^2+b^2-ab \\ 0 & c-b & c^2-b^2+ab-ac \\ 1 & d-b & -b^2+d^2+ad-ad \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & b+a & a^2+b^2-ab \\ 0 & 1 & c+b-a \\ 1 & 1 & b+d-a \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & c+b-a \\ 1 & d+b-a \end{vmatrix}$$

$$\underline{\underline{= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)}}$$

Q(12). Show that :

$$\begin{pmatrix} (x-a)^2 & (y-a)^2 & (z-a)^2 \\ (x-b)^2 & (y-b)^2 & (z-b)^2 \\ (x-c)^2 & (y-c)^2 & (z-c)^2 \end{pmatrix} = \begin{pmatrix} 1 & -2a & a^2 \\ 1 & -2b & b^2 \\ 1 & -2c & c^2 \end{pmatrix}\begin{pmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix}$$

Now use property of determinant that $|AB| = |A||B|$ and $|A| = |A^T|$, $|B| = |B^T|$

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix}$$ and for the second determinant

$$\begin{vmatrix} 1 & -2a & a^2 \\ 1 & -2b & b^2 \\ 1 & -2c & c^2 \end{vmatrix} = -2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad C_3 \longleftrightarrow C_1$$

$$\begin{vmatrix} 1 & -2a & a^2 \\ 1 & -2b & b^2 \\ 1 & -2c & c^2 \end{vmatrix} = -2\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 2\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

Therefore, $$\begin{vmatrix} (x-a)^2 & (y-a)^2 & (z-a)^2 \\ (x-b)^2 & (y-b)^2 & (z-b)^2 \\ (x-c)^2 & (y-c)^2 & (z-c)^2 \end{vmatrix} = \begin{vmatrix} 1 & -2a & a^2 \\ 1 & -2b & b^2 \\ 1 & -2c & c^2 \end{vmatrix}\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \times \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

Q(13). Prove that

$$\begin{vmatrix} a^2+p & ab & ac & ad \\ ab & b^2+p & bc & bd \\ ac & bc & c^2+p & cd \\ ad & bd & bc & d^2+p \end{vmatrix} = p^3(a^2+b^2+c^2+d^2+p)$$

Multiply each row by a, b, c and d respectively and divide with the same, then

$$=\frac{1}{abcd}\begin{vmatrix} a^3+pa & a^2b & a^2c & a^2d \\ ab^2 & b^3+pb & b^2c & b^2d \\ c^2a & c^2b & c^3+pc & c^2d \\ d^2a & d^2b & d^2c & d^3+pd \end{vmatrix}$$ Takeout factor a, b, c and d from the columns 1 ,2, 3, and 4.

$$=\frac{1}{abcd}\begin{vmatrix} a^3+pa & a^2b & a^2c & a^2d \\ ab^2 & b^3+pb & b^2c & b^2d \\ c^2a & c^2b & c^3+pc & c^2d \\ d^2a & d^2b & d^2c & d^3+pd \end{vmatrix} = \begin{vmatrix} a^2+p & a^2 & a^2 & a^2 \\ b^2 & b^2+p & b^2 & b^2 \\ c^2 & c^2 & c^3+p & c^2 \\ d^2 & d^2 & d^2 & d^2+p \end{vmatrix}$$

$R_1 \to R_1 + R_2 + R_3 + R_4$

$$=(p+a^2+b^2+c^2+d^2)\begin{vmatrix} 1 & 1 & 1 & 1 \\ b^2 & b^2+p & b^2 & b^2 \\ c^2 & c^2 & c^3+p & c^2 \\ d^2 & d^2 & d^2 & d^2+p \end{vmatrix}$$

$C_4 \to -C_1 + C_4$
$C_3 \to -C_1 + C_3$
$C_2 \to -C_1 + C_2$

$$=(p+a^2+b^2+c^2+d^2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ b^2 & p & 0 & 0 \\ c^2 & 0 & p & 0 \\ d^2 & 0 & 0 & p \end{vmatrix}$$

$$=\underline{\underline{(1+a^2+b^2+c^2+d^2)}}p^3$$

Q(14). Let a_{ij} be an i-j element of determinant Dn of order nxn, define as follows

$$a_{ij} = a\,; \quad \text{if} \quad i < j$$

$$a_{ij} = x\,; \quad \text{if} \quad i = j$$

$$a_{ij} = -a\,; \quad \text{if} \quad i > j$$

$$D_n = \begin{vmatrix} x & a & a & ..... & .... & a \\ -a & x & a & a & ... & a \\ -a & -a & x & a & .... & a \\ .... & .... & ... & ... & ..... & .... \\ -a & -a & -a & -a & x & a \\ -a & -a & -a & -a & -a & x \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$D_n = \begin{vmatrix} x-a & 0 & 0 & ..... & .... & a+x \\ -a & x & a & a & ... & a \\ -a & -a & x & a & .... & a \\ .... & .... & ... & ... & ..... & .... \\ -a & -a & -a & -a & x & a \\ -a & -a & -a & -a & -a & x \end{vmatrix}_n$$

$$= (x-a)D_{n-1} + (a+x)\begin{vmatrix} -a & x & a & ..... & .... & a \\ -a & x & a & a & ... & a \\ -a & -a & x & a & .... & a \\ .... & .... & ... & ... & ..... & .... \\ -a & -a & -a & -a & -a & x \\ -a & -a & -a & -a & -a & -a \end{vmatrix}_{n-1}$$

$R_{n-2} \to -R_{n-1} + R_{n-2}$

$R_{n-3} \to -R_{n-1} + R_{n-34}$

$R_{n-2} \to -R_{n-1} + R_{n-4}$

....

$R_2 \to -R_{n-2} + R_2$

$$= (x-a)D_{n-1} + (a+x)\begin{vmatrix} 0 & a+x & 2a & ..... & .... & 2a \\ 0 & a+x & 2a & 2a & ... & 2a \\ 0 & 0 & a+x & 2a & .... & 2a \\ .... & .... & ... & ... & ..... & .... \\ 0 & 0 & 0 & 0 & 0 & a+x \\ -a & -a & -a & -a & -a & -a \end{vmatrix}_{n-1}$$

$$D_n = (x-a)D_{n-1} + (a+x)^{n-1}a$$

Now use method of induction to prove the given statement

When n =2 $D_2 = \begin{vmatrix} x & a \\ -a & x \end{vmatrix} = x^2 + a^2$

And $D_2 = (x-a)D_1 + (a+x)^1 a = (x-a)x + (a+x) = x^2 + a^2$

Next assume that it is true up to n = p

Then, $D_p = (x-a)D_{p-1} + (a+x)^{p-1}a$

When n = p+1

$$D_{p+1} = (x-a)D_p + (a+x)^p a = \frac{1}{2}\left[(x-a)^p + (x+a)^p\right](x-a) + (x+a)^p a$$

$$= \frac{1}{2}\left[(x-a)^{p+1}\right] + \frac{1}{2}\left[(x+a)^{p+1}\right]$$

Therefore it is rue for every positive integer n

Hence, $\underline{\underline{D_n = \frac{1}{2}\left[(x-a)^n + (x+a)^n\right]}}$

Q(15).

$$\begin{vmatrix} a_1+1 & a_1 & a_1 & .. & a_1 \\ a_2 & a_2+1 & a_2 & .. & a_2 \\ a_3 & a_3 & a_3+1 & .. & a_3 \\ .. & .. & .. & .. & .. \\ a_n & a_n & a_n & .. & a_n+1 \end{vmatrix}_n \quad R_1 \to R_1+R_2+R_3+....+R_n$$

$$\begin{vmatrix} a_n+a_2+a_1+1 & a_n..+a_2+a_1+1 & a_n+a_2+a_1+1 & .. & a_n.+a_2+a_1+1 \\ a_2 & a_2+1 & a_2 & .. & a_2 \\ a_3 & a_3 & a_3+1 & .. & a_3 \\ .. & .. & .. & .. & .. \\ a_n & a_n & a_n & .. & a_n+1 \end{vmatrix}_n$$

$$\begin{vmatrix} 1 & 1 & 1 & .. & 1 \\ a_2 & a_2+1 & a_2 & .. & a_2 \\ a_3 & a_3 & a_3+1 & .. & a_3 \\ .. & .. & .. & .. & .. \\ a_n & a_n & a_n & .. & a_n+1 \end{vmatrix}(1+a_1+a_2..+a_n)$$

$$\begin{matrix} C_1 \to -C_n+C_1 \\ C_2 \to -C_n+C_2 \\ C_3 \to -C_n+C_3 \\ .... \\ C_{n-1} \to -C_n+C_{n-1} \end{matrix} \begin{vmatrix} 0 & 0 & 0 & .. & 1 \\ 0 & 1 & 0 & .. & a_2 \\ 0 & 0 & 1 & .. & a_3 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & a_n+1 \end{vmatrix}(1+a_1+a_2..+a_n)$$

$$= (1+a_1+a_2..+a_n)$$

Q(16). Prove that

$$\begin{vmatrix} a & b & b & .... & b \\ b & a & b & .... & b \\ b & b & a & ... & b \\ .. & .... & .... & .... & .... \\ b & b & b & ..... & a \end{vmatrix} \qquad \begin{array}{l} C_1 \to -C_n + C_1 \\ C_2 \to -C_n + C_2 \\ C_3 \to -C_n + C_3 \\ .... \\ C_{n-1} \to -C_n + C_{n-1} \end{array}$$

$$\begin{vmatrix} a-b & 0 & 0 & .... & b \\ 0 & a-b & 0 & .... & b \\ 0 & 0 & a-b & ... & b \\ .. & .... & .... & .... & .... \\ b-a & b-a & b-a & ..... & a \end{vmatrix}$$

$$= (a-b)^{n-1} \begin{vmatrix} 1 & 0 & 0 & .... & b \\ 0 & 1 & 0 & .... & b \\ 0 & 0 & 1 & ... & b \\ .. & .... & .... & .... & .... \\ -1 & -1 & -1 & ..... & a \end{vmatrix} R_n \to R_1 + R_2 + R_3 + ... + R_n$$

$$= (a-b)^{n-1} \begin{vmatrix} 1 & 0 & 0 & .... & b \\ 0 & 1 & 0 & .... & b \\ 0 & 0 & 1 & ... & b \\ .. & .... & .... & .... & .... \\ 0 & 0 & 0 & ..... & a+(n-)b \end{vmatrix} \text{Expand through nth row}$$

$$= (a-b)^{n-1}(a+(n-1)b) \begin{vmatrix} 1 & 0 & 0 & .... & 0 \\ 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 1 & ... & 0 \\ .. & .... & .... & .... & .... \\ 0 & 0 & 0 & ..... & 1 \end{vmatrix}$$

$$= (a-b)^{n-1}(a+(n-1)b)$$

Q(17). Show that

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} \qquad \begin{matrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to cR_3 \\ R_4 \to dR_4 \end{matrix} \text{ and divide by the same}$$

$$= \frac{1}{abcd}\begin{vmatrix} a^2 & -ab & -a^2 & ab \\ b^2 & ab & -b^2 & -ab \\ c^c & -cd & c^2 & -dc \\ d^2 & cd & d^2 & cd \end{vmatrix} \quad \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + R_4 \end{matrix}$$

$$= \frac{1}{abcd}\begin{vmatrix} a^2+b^2 & 0 & -a^2-b^2 & 0 \\ b^2 & ab & -b^2 & -ab \\ c^2+d^2 & 0 & c^2+d^2 & 0 \\ d^2 & cd & d^2 & cd \end{vmatrix}$$

$$= \frac{1}{abcd}(a+b^2)(c^2+d^2)\begin{vmatrix} 1 & 0 & -1 & 0 \\ b^2 & ab & -b^2 & -ab \\ 1 & 0 & 1 & 0 \\ d^2 & cd & d^2 & cd \end{vmatrix}$$

$$= \frac{1}{abcd}(a+b^2)(c^2+d^2)bd\begin{vmatrix} 1 & 0 & -1 & 0 \\ b & a & -b & -a \\ 1 & 0 & 1 & 0 \\ d & c & 2d & c \end{vmatrix} \quad C_3 \to C_1 + C_3$$

$$= \frac{1}{abcd}(a+b^2)(c^2+d^2)bd\begin{vmatrix} 1 & 0 & 0 & 0 \\ b & a & 0 & -a \\ 1 & 0 & 2 & 0 \\ d & c & 2d & c \end{vmatrix} \text{Expand through first row}$$

$$= \frac{2}{abcd}(a+b^2)(c^2+d^2)abd\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ c & 2d & c \end{vmatrix} \text{ Expand through second row}$$

$$= \frac{2}{abcd}(a+b^2)(c^2+d^2)abd(2c)$$

$$= \underline{\underline{4(a+b^2)(c^2+d^2)}}$$

Q(17). D_n is the nth order determinant

$$D_n = \begin{vmatrix} 1+x^2 & x & 0 & .... & 0 \\ x & 1+x^2 & x & .... & 0 \\ 0 & x & 1+x^2 & ..... & 0 \\ ... & .... & x & 1+x^2 & x \\ 0 & 0 & 0 & x & 1+x^2 \end{vmatrix}_n \quad \text{Expand through last row,}$$

$$D_n = (1+x^2)\begin{vmatrix} 1+x^2 & x & 0 & .... & 0 \\ x & 1+x^2 & x & .... & 0 \\ 0 & x & 1+x^2 & ..... & 0 \\ ... & .... & x & 1+x^2 & x \\ 0 & 0 & 0 & x & 1+x^2 \end{vmatrix}_{n-1}$$

$$-x\begin{vmatrix} 1+x^2 & x & 0 & .... & 0 \\ x & 1+x^2 & x & .... & 0 \\ 0 & 0 & 0 & x & 0 \\ ... & .... & 0 & 1+x^2 & 0 \\ 0 & 0 & 0 & x & x \end{vmatrix}_{n-1}$$

Expand second determinant through the last column, we get

$$D_n = (1+x^2)D_{n-1} - x^2\begin{vmatrix} 1+x^2 & x & 0 & .... & 0 \\ x & 1+x^2 & x & .... & 0 \\ 0 & 0 & 0 & x & 0 \\ ... & .... & x & 1+x^2 & x \\ 0 & 0 & 0 & x & 1+x^2 \end{vmatrix}_{n-2}$$

$$\underline{\underline{D_n = (1+x^2)D_{n-1} - x^2 D_{n-2}}}$$

when x = 1 $D_2 = 3$, $D_4 = 4$ and use the recurrence relation $D_n = 2D_{n-1} - D_{n-2}$

Hence $D_{10} = 11$

Q(18).

$$\begin{vmatrix} 0 & 1 & 2 & 3 & 4 & .... & n-1 \\ 1 & 0 & 1 & 2 & 3 & .... & n-2 \\ 2 & 1 & 0 & 1 & 2 & .... & n-3 \\ 3 & 2 & 1 & 0 & 1 & .... & n-4 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ n-2 & n-3 & n-4 & n-5 & ... & .... & 1 \\ n-1 & n-2 & n-3 & n-4 & & 1 & 0 \end{vmatrix} \quad C_1 \to -C_2 + C_1$$

$$\begin{vmatrix} -1 & 1 & 2 & 3 & 4 & .... & n-1 \\ 1 & 0 & 1 & 2 & 3 & .... & n-2 \\ 1 & 1 & 0 & 1 & 2 & .... & n-3 \\ 1 & 2 & 1 & 0 & 1 & .... & n-4 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ 1 & n-3 & n-4 & n-5 & ... & .... & 1 \\ 1 & n-2 & n-3 & n-4 & & 1 & 0 \end{vmatrix}$$ Similar column operations can be used for

remaining columns, then we get

$$\begin{vmatrix} -1 & -1 & -1 & -1 & -1 & .... & n-1 \\ 1 & -1 & -1 & -1 & -1 & .... & n-2 \\ 1 & 1 & -1 & -1 & -1 & .... & n-3 \\ 1 & 1 & 1 & -1 & -1 & .... & n-4 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ 1 & 1 & 1 & 1 & ..1. & .... & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix} \quad R_n \to R_1 + R_n$$

$$D_n = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & .... & n-1 \\ 1 & -1 & -1 & -1 & -1 & .... & n-2 \\ 1 & 1 & -1 & -1 & -1 & .... & n-3 \\ 1 & 1 & 1 & -1 & -1 & .... & n-4 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ 1 & 1 & 1 & 1 & ..1. & .... & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

Expand the determinant through the first row, we get

$$D_n = (n-1)\begin{vmatrix} 1 & -1 & -1 & -1 & -1 & .... & -1 \\ 1 & 1 & -1 & -1 & -1 & .... & -1 \\ 1 & 1 & 1 & -1 & -1 & .... & -1 \\ 1 & 1 & 1 & 1 & -1 & .... & -1 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ 1 & 1 & 1 & 1 & ..1. & .... & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}_{n-1} \quad \begin{array}{l} R_1 \to R_n + R_1 \\ R_2 \to R_n + R_2 \\ .... \\ R_{n-2} \to R_n + R_{n-2} \end{array}$$

$$D_n = (n-1)\begin{vmatrix} 1 & -2 & -2 & -2 & -2 & .... & -2 \\ 1 & 0 & -2 & -2 & -2 & .... & -2 \\ 1 & 0 & 0 & -2 & -2 & .... & -2 \\ 1 & 0 & 0 & 0 & 0 & .... & -2 \\ ... & ... & .... & ..... & ..... & .... & ..... \\ 1 & 0 & 0 & 0 & -2 & .... & -2 \\ 1 & 0 & 0 & 0 & 0 & ..... & 2 \end{vmatrix}_{n-1}$$

$$\underline{\underline{D_n = (n-1)(-2)^{n-2}}}$$