MODEL QUESIONS & ANSWERS MA(101)

VECTORS

Exercice 1

Q(1). (a) $\overrightarrow{OP} = \underline{i} + 3\underline{j} + 4\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(\underline{i} + 3\underline{j} + 4\underline{k}) \cdot (\underline{i} + 3\underline{j} + 4\underline{k})} = \sqrt{(1^2 + 3^2 + 4^2)} = \sqrt{26}$

(b) $\overrightarrow{OP} = 2\underline{i} + 4\underline{j} + 5\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(2\underline{i} + 4\underline{j} + 5\underline{k}) \cdot (2\underline{i} + 4\underline{j} + 5\underline{k})} = \sqrt{(2^2 + 4^2 + 5^2)} = \sqrt{45}$

(c) $\overrightarrow{OP} = 4\underline{i} + 0\underline{j} + 2\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(4\underline{i} + 0\underline{j} + 2\underline{k}) \cdot (4\underline{i} + 0\underline{j} + 2\underline{k})} = \sqrt{(4^2 + 0^2 + 2^2)} = \sqrt{20}$

Q(2). (a) $\overrightarrow{OP} = 2\underline{i} - \underline{j} - \underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(2^2 + 1^2 + 1^2)} = \sqrt{6}$, $\frac{\overrightarrow{OP}}{|OP|} = \frac{2\underline{i} - \underline{j} - \underline{k}}{\sqrt{6}}$, then

$\cos\theta_1 = \frac{2}{\sqrt{6}}, \cos\theta_2 = \frac{-1}{\sqrt{6}}$ and $\cos\theta_3 = \frac{-1}{\sqrt{6}}$.

(b) $\overrightarrow{OP} = 4\underline{i} + 0\underline{j} + 2\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(4^2 + 0^2 + 2^2)} = \sqrt{20}$, $\frac{\overrightarrow{OP}}{|OP|} = \frac{4\underline{i} + 0\underline{j} + 2\underline{k}}{\sqrt{20}}$, then

$\cos\theta_1 = \frac{4}{\sqrt{20}}, \cos\theta_2 = 0$ and $\cos\theta_3 = \frac{2}{20}$

(c) $\overrightarrow{OP} = -\underline{i} + 2\underline{j} + 1\underline{k}$ $\left|\overrightarrow{OP}\right| = \sqrt{(1^2 + 2^2 + 1^2)} = \sqrt{6}$, $\frac{\overrightarrow{OP}}{|OP|} = -\frac{-\underline{i} + 2\underline{j} + \underline{k}}{\sqrt{6}}$, then

$\cos\theta_1 = \frac{-1}{\sqrt{6}}, \cos\theta_2 = \frac{2}{\sqrt{6}}$ and $\cos\theta_3 = \frac{1}{\sqrt{6}}$

Q(3). (a) $\overrightarrow{OP} = \underline{i} + \underline{j} + \underline{k}$ then direction ratios line OP is $1:1:1$

$\left|\overrightarrow{OP}\right| = \sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$, $\frac{\overrightarrow{OP}}{|OP|} = -\frac{\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$, then direction cosine of line OP :

$\cos\theta_1 = \frac{1}{\sqrt{3}}, \cos\theta_2 = \frac{1}{\sqrt{3}}$ and $\cos\theta_3 = \frac{1}{\sqrt{3}}$

(b) $\overrightarrow{OP} = -\underline{i} + \underline{j} + \underline{k}$ then direction ratios line OP is $-1:1:1$

$\left|\overrightarrow{OP}\right| = \sqrt{(1^2+1^2+1^2)} = \sqrt{3}$, $\dfrac{\overrightarrow{OP}}{|OP|} = -\dfrac{-\underline{i}+\underline{j}+\underline{k}}{\sqrt{3}}$, then direction cosine of line OP :

$\cos\theta_1 = \dfrac{-1}{\sqrt{3}}, \cos\theta_2 = \dfrac{1}{\sqrt{3}}$ and $\cos\theta_3 = \dfrac{1}{\sqrt{3}}$

(c) $\overrightarrow{OP} = 2\underline{i}+\underline{j}-\underline{k}$ then direction ratios line OP is $2:1:-1$

$\left|\overrightarrow{OP}\right| = \sqrt{(2^2+1^2+1^2)} = \sqrt{6}$, $\dfrac{\overrightarrow{OP}}{|OP|} = \dfrac{2\underline{i}+\underline{j}-\underline{k}}{\sqrt{6}}$, then direction cosine of line OP :

$\cos\theta_1 = \dfrac{2}{6}, \cos\theta_2 = \dfrac{1}{\sqrt{6}}$ and $\cos\theta_3 = \dfrac{-1}{\sqrt{6}}$

Q(4). Determine the angles $(\theta_1, \theta_2, \theta_3)$ for the vectors with the direction cosines:

(a) $\left\{\dfrac{\sqrt{3}}{2}, 0, \dfrac{1}{2}\right\}, \Rightarrow \quad \cos\alpha = \dfrac{\sqrt{3}}{2}, \cos\beta = 0, \quad \cos\gamma = \dfrac{1}{2}$

$\alpha = \pi/6$, $\beta = \pi/2$ and $\gamma = \pi/3$

(b) $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right\} \Rightarrow \alpha = \beta = \gamma = \cos^{-1}\dfrac{1}{\sqrt{3}}$

(c) $\left\{\dfrac{1}{3}, -\dfrac{1}{3}, \dfrac{\sqrt{7}}{3}\right\} \Rightarrow \quad \alpha = \cos^{-1}\dfrac{1}{3}, \quad \beta = \cos^{-1}-\dfrac{1}{3}, \quad \gamma = \cos^{-1}\dfrac{\sqrt{7}}{3}$

5. Determine the lengths $|\overrightarrow{AB}|$ of the vectors $\underline{AB}$, given that the end points A and B. Use your results to determine the direction cosines for each of these vectors.

(a) A=(1,1,1), B= (2,O,6)

$\overrightarrow{AB} = \underline{i}-\underline{j}+5\underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{1^2+1^2+5^2} = \sqrt{27}.$

$\cos\alpha = \dfrac{1}{\sqrt{27}}, \cos\beta = \dfrac{-1}{\sqrt{27}}, \cos\gamma = \dfrac{5}{\sqrt{27}},$

(b) A=(2,-1,1), B=(-2,2,2)

$\overrightarrow{AB} = -4\underline{i} + \underline{j} + \underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{4^2 + 1^2 + 1^2} = \sqrt{18}.$

$\cos\alpha = \frac{-4}{\sqrt{18}}, \ \cos\beta = \frac{1}{\sqrt{18}}, \cos\gamma = \frac{1}{\sqrt{18}},$

(c) A=(-1,3,1), B= (-2,-1,0).

$\overrightarrow{AB} = -\underline{i} - 4\underline{j} - 1\underline{k}$ and $\left|\overrightarrow{AB}\right| = \sqrt{1^2 + 4^2 + 1^2} = \sqrt{18}.$

$\cos\alpha = \frac{-1}{\sqrt{18}}, \ \cos\beta = \frac{4}{\sqrt{18}}, \cos\gamma = \frac{-1}{\sqrt{18}},$

Q(6). Write down the position vectors OP in terms of the unit vectors i, j, k given that 0 is the origin and the points P are:

(a) (1,1,1) $\overrightarrow{OP} = \underline{i} + \underline{j} + \underline{k}$, (b) (2,3,4) $\overrightarrow{OP} = 2\underline{i} + 3\underline{j} + 4\underline{k}$, (c) (1,2,3) $\overrightarrow{OP} = \underline{i} + 2\underline{j} + 3\underline{k}$,

Q(7). Determine the values of α, β and γ in order to that:

$$(1-\alpha)\underline{i} + \beta(1-\alpha^2)\underline{j} + (\gamma - 2)\underline{k} = \frac{1}{2}\underline{i} + 3\underline{j} + 2\underline{k}$$

Equating i, j and k components

$$(1-\alpha) = \frac{1}{2} \Rightarrow \ \alpha = \frac{1}{2} \ \beta(1-\alpha^2) = 3 \Rightarrow \ \beta = 4 \ \text{and} \ (\gamma - 2) = 2, \ \Rightarrow \ \gamma = 4$$

Q(8). Form the sum $\underline{a} + \underline{b}$ and difference $\underline{a} - \underline{b}$ of the vectors:

(a) $\underline{a} = 3\underline{i} + 3\underline{j} + 2\underline{k} \quad \underline{b} = \underline{i} + \underline{j} + 2\underline{k}$

$\Rightarrow \ \underline{a} + \underline{b} = 4\underline{i} + 4\underline{j} + 4\underline{k} \ \Rightarrow \ \underline{a} - \underline{b} = 2\underline{i} + 2\underline{j}$

(b) $\underline{a} = -\underline{i} + 2\underline{j} - 2\underline{k} \quad \underline{b} = \underline{i} - \underline{j} + 2\underline{k}$

$\Rightarrow \ \underline{a} + \underline{b} = \underline{j} \Rightarrow \ \underline{a} - \underline{b} = -2\underline{i} + 3\underline{j} - 4\underline{k}$

(c) $\underline{a} = 3\underline{i} + \underline{j} + 2\underline{k} \quad \underline{b} = \underline{i} + 3\underline{j} + 2\underline{k}$

$\Rightarrow \ \underline{a} + \underline{b} = 4\underline{i} + 4\underline{j} + 4\underline{k} \Rightarrow \ \underline{a} - \underline{b} = 2\underline{i} + 2\underline{j}$

Q(9). State which of the following pairs of vectors a and b are parallel and which are anti-parallel:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k} \quad \underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k} \ \Rightarrow \ \underline{b} = -4\underline{a}, \ \underline{b} // \underline{a},$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k} \quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \ \Rightarrow \ \underline{b} = -\underline{a}, \ \underline{b} // \underline{a},$

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k} \quad \underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k} \ \Rightarrow$ $\underline{b}$ and $\underline{a}$, are anti parallel.

Q(10). Express the following vectors $\underline{a}$ as the product of a scalar and a unit vector:

(a) $\underline{a} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{14}}$, $\underline{a} = \sqrt{14}$ times unit vector

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k} \Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{-2\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{14}}$, $\underline{a} = \sqrt{14}$ times unit vector

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k} \Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{4\underline{i} - \underline{j} - 3\underline{k}}{\sqrt{26}}$, $\underline{a} = \sqrt{26}$ times unit vector

Q(11). Find the vectors $\underline{AB}$, and their direction cosines given that A and B have position vectors a and b, respectively, where

(a) $\underline{a} = \underline{i} - 3\underline{j} + 2\underline{k}$ $\underline{b} = -\underline{i} + \underline{j} - 4\underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = -2\underline{i} + 4\underline{j} - 6\underline{k}$ direction cosines of AB is $-2:4:-6$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = 4\underline{i} - 4\underline{j} + 2\underline{k}$ direction cosines of AB is $4:-4:2$

(c) $\underline{a} = 4\underline{i} - \underline{j} + \underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{AB} = \underline{b} - \underline{a} = -\underline{i} + 3\underline{j} + 2\underline{k}$ direction cosines of AB is $-1:3:2$

Q(12). Find the scalar products $\underline{a} \cdot \underline{b}$ and hence find the angle between the vectors $\underline{a}$ and $\underline{b}$ given that:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k} \Rightarrow \underline{a} \cdot \underline{b} = -4 - 36 - 4 = -44$

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \underline{a} \cdot \underline{b} = -4 - 9 - 4 = -44$

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ $\underline{b} = 3\underline{i} + 2\underline{j} + 3\underline{k} \Rightarrow \underline{a} \cdot \underline{b} = 12 - 2 - 9 = 1$

13 Find unit vectors parallel to the vectors a where:

(a) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ Unit vector parallel to $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ is $\Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{\underline{i} - 3\underline{j} + \underline{k}}{\sqrt{11}}$,

(b) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ Unit vector parallel to $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ is $\Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{-2\underline{i} + 3\underline{j} - \underline{k}}{\sqrt{14}}$,

(c) $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ Unit vector parallel to $\underline{a} = 4\underline{i} - \underline{j} - 3\underline{k}$ is $\Rightarrow \frac{\underline{a}}{|\underline{a}|} = \frac{4\underline{i} - \underline{j} - 3\underline{k}}{\sqrt{26}}$,

Q(14). Evaluate the vector products $\underline{b} x \underline{a}$ given that

(a) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k} \Rightarrow \underline{b} x \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{vmatrix} = 0$

(b) $\underline{a} = -\underline{i} + \underline{j} + \underline{k}$ $\quad \underline{b} = 2\underline{i} + 4\underline{j} + 3\underline{k} \Rightarrow \underline{b} x \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 4 & 3 \\ -1 & 1 & 1 \end{vmatrix} \Rightarrow \underline{b} x \underline{a} = \underline{i} - 5\underline{j} - 6\underline{k}$

(c) $\underline{a} = -\underline{i} - \underline{j} + \underline{k}$ $\quad \underline{b} = 2\underline{i} + 2\underline{j} + 2\underline{k} \Rightarrow \underline{b} x \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2 & 2 \\ -1 & -1 & 1 \end{vmatrix} \Rightarrow \underline{b} x \underline{a} = 4\underline{i} - 4\underline{j}$

Q(15). Evaluate the triple scalar products $\underline{a} \cdot (\underline{b} x \underline{c})$ and $\underline{b} \cdot (\underline{a} x \underline{c})$ given that:

(a) $\underline{a} = -2\underline{i} + 3\underline{j} - \underline{k}$ $\quad \underline{b} = 2\underline{i} - 3\underline{j} + \underline{k}$ and $\underline{c} = 4\underline{i} - \underline{j} - 3\underline{k}$

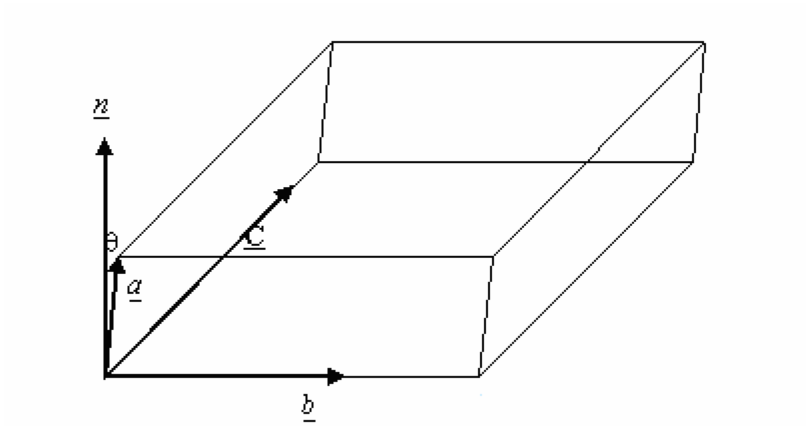
$$\Rightarrow \underline{a} \cdot (\underline{b} x \underline{c}) = \begin{vmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ 4 & -1 & -3 \end{vmatrix} = 0 \Rightarrow \underline{b} \cdot (\underline{a} x \underline{c}) = \begin{vmatrix} 2 & -3 & 1 \\ -2 & 3 & -1 \\ 4 & -1 & -3 \end{vmatrix} = 0$$

(b) $\underline{a} = \underline{i} - 3\underline{j} + \underline{k}$ $\quad \underline{b} = -4\underline{i} + 12\underline{j} - 4\underline{k}$ and $\underline{c} = 2\underline{i} + 2\underline{j} + 2\underline{k}$

$$\Rightarrow \underline{a} \cdot (\underline{b} x \underline{c}) = \begin{vmatrix} 1 & -3 & 1 \\ -4 & 12 & -4 \\ 2 & 2 & 2 \end{vmatrix} = 0 \Rightarrow \underline{b} \cdot (\underline{a} x \underline{c}) = \begin{vmatrix} -4 & 12 & -4 \\ 1 & -3 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 0$$

Q(16). The scalar quantity $\underline{a}.(\underline{b} \times \underline{c})$ is known as the scalar triple product of $\underline{a}$ and $\underline{b} x \underline{c}$. It is often denoted by $[\underline{a}, \underline{b}, \underline{c}]$.

The magnitude of this quantity is the volume of the parallelopied formed by the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$, i.e. $|\underline{a}.(\underline{b}\times\underline{c})| = |\underline{a}||\underline{b}\mathrm{x}\underline{c}|\cos\theta$, θ being the angle between $\underline{a}$, and $\underline{b}\mathrm{x}\underline{c}$



Let $\underline{a}$, $\underline{b}$, $\underline{c}$ be three given vectors. We can permute the three given vectors in six different ways. Also each manner of writing down the three vectors gives rise to two scalar triple products depending upon the positions of dot and cross. Thus, we have the following twelve scalar triple products

$(\underline{a}\times\underline{b})\cdot\underline{c}$,$(\underline{b}\times\underline{c})\cdot\underline{a}$, $(\underline{c}\times\underline{a})\cdot\underline{b}$ $(\underline{a}\cdot\underline{b})\times\underline{c}$,$(\underline{b}\cdot\underline{c})\times\underline{a}$, $(\underline{c}\cdot\underline{a})\times\underline{b}$

$(\underline{a}\times\underline{c})\cdot\underline{b}$,$(\underline{b}\times\underline{a})\cdot\underline{c}$, $(\underline{c}\times\underline{b})\cdot\underline{a}$, $(\underline{a}\cdot\underline{c})\times\underline{b}$, $(\underline{b}\cdot\underline{a})\times\underline{c}$,$(\underline{c}\cdot\underline{b})\times\underline{a}$

We shall now prove the following two important results

(i). A cyclic permutation of three vectors does not change the value of the scalar triple product and an anti-cyclic permutation changes the value in sign but not in magnitude.

(ii). The positions of dot and cross can be interchanged without any change in the value of the scalar triple product.

Firstly suppose that $\underline{a}$, $\underline{b}$, $\underline{c}$ is a right-handed system so have $V = (\underline{a}\times\underline{b})\cdot\underline{c}$.

The vector triads $\underline{b}, \underline{c}, \underline{a}$ and $\underline{b}, \underline{c}, \underline{a}$ are also right-handed and the parallelopiped with OA, OB, OC as adjacent edges is the same as that with OB, OC, OA or with OC, OA, OB as adjacent edges. Thus,

$$V = [\underline{b}, \underline{c}, \underline{a}] \text{ and } V = [\underline{c}, \underline{a}, \underline{b}]$$

1.If $(\underline{a}x\underline{b}).\underline{c} = 0$ since $(\underline{a} \times \underline{b})$ is perpendicular to both $\underline{a}$, and $\underline{b}$, then vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are coplanar.

2. For the nonzero vectors $\underline{a}, \underline{b}$, and $\underline{c}$, they are coplanar (ie lie on the same plane) if and only if $[\underline{a}, \underline{b}, \underline{c}]=0$.

Q(17). If $\underline{A} = 2\underline{i} + 3\underline{j} - 4\underline{k}$, $\underline{B} = 3\underline{i} + 5\underline{j} + 2\underline{k}$ and $\underline{C} = \underline{i} - 2\underline{j} + 3\underline{k}$ determine $\underline{A}.\underline{B}$, $\underline{A}x\underline{B}$ and $\underline{A}.(\underline{B}x\underline{C})$.

$$\underline{A} \cdot \underline{B} = (2\underline{i} + 3\underline{j} - 4\underline{k}) \cdot (3\underline{i} + 5\underline{j} + 2\underline{k}) = 13$$

$$\Rightarrow \underline{A}x\underline{B} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 4 \\ 3 & 5 & 2 \end{vmatrix} = -14\underline{i} + 8\underline{j} + 4\underline{k}$$

$$\Rightarrow \underline{A}.(\underline{B}x\underline{C}) = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 5 & 2 \\ 1 & -2 & 3 \end{vmatrix} = -27$$

Q(18). If $\underline{A} = \underline{i} + 3\underline{j} + 5\underline{k}$, $\underline{B} = 3\underline{i} + \underline{j} + 2\underline{k}$ and $\underline{C} = \underline{i} - \underline{j} + \underline{k}$ find $\underline{A}x(\underline{B}x\underline{C})$ and $(\underline{A}x\underline{B})x\underline{C}$.

Use the expansion $\underline{A}x(\underline{B}x\underline{C}) = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$

$(\underline{A} \cdot \underline{C}) = 3, \quad (\underline{A} \cdot \underline{B}) = 16,$

$\Rightarrow \underline{A}x(\underline{B}x\underline{C}) = 3(3\underline{i} + \underline{j} + 2\underline{k}) - 16(\underline{i} - \underline{j} + \underline{k}) = (-7\underline{i} + 19\underline{j} - 10\underline{k})$

$(\underline{A}x\underline{B})x\underline{C} = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{B} \cdot \underline{C})\underline{A}$

$\Rightarrow (\underline{A}x\underline{B})x\underline{C} = 3(3\underline{i} + \underline{j} + 2\underline{k}) - 4(\underline{i} + 3\underline{j} + 5\underline{k}) = (5\underline{i} - 9\underline{j} - 14\underline{k})$

Q(19). If $\underline{F} = x^2\underline{i} + x^4\underline{j} + 2x\underline{k}$ then $\frac{d\underline{F}}{dx} = 2x\underline{i} + 4x^3\underline{j} + 2\underline{k}$ and $\frac{d^2\underline{F}}{dx^2} = 2\underline{i} + 12x^2$

$$\left|\frac{d\underline{F}}{dx}\right| = \sqrt{4x^2 + 16x^6 + 4}$$

Q(20). Find the unit normal vector to the surface $\phi = xz^2 + 3xy - 2yz^2 + 1 = 0$ at the point (1,-2,-1)

$$\nabla\phi = (z^2 + 3y)\underline{i} + (3x - 2z^2)\underline{j} + (2xz - 4yz)\underline{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = -5\underline{i} + \underline{j} - 10\underline{k}$$

$$\underline{n} = \frac{(\nabla\phi)}{|(\nabla\phi)|} = \frac{-5\underline{i} + \underline{j} - 10\underline{k}}{\sqrt{62}}$$

Q(21). Determine the directional derivative of $\phi = xe^y + yz^2 + xyz$ at the point (2,0,3) in the direction $\underline{A} = 3\underline{i} - 2\underline{j} + \underline{k}$.

Directional derivative at a given direction is defined by $\nabla\phi \cdot \underline{n}$

$$\nabla\phi = (e^y + yz)\underline{i} + (xe^y + z^2 + xz)\underline{j} + (2yz + xz)\underline{k}$$

$$(\nabla\phi)_{(2,0,3)} = \underline{i} + 17\underline{j} \Rightarrow \quad (\nabla\phi)\cdot\underline{n} = (\underline{i} + 17\underline{j})\cdot\frac{(3\underline{i} - 2\underline{j} + \underline{k})}{\sqrt{14}} = \underline{\underline{\frac{-31}{\sqrt{14}}}}$$

Q(22). Determine the values of P such that the three vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$ are coplanar, when $\underline{A} = 2\underline{i} + \underline{j} + 4\underline{k}$, $\underline{B} = 3\underline{i} + 2\underline{j} + P\underline{k}$ and $\underline{C} = \underline{i} + 4\underline{j} + 2\underline{k}$.

When vectors $\underline{A}$, $\underline{B}$, and $\underline{C}$ are coplanar, then $(\underline{A} \times \underline{B}) \cdot \underline{C} = 0$

$$(\underline{A} \times \underline{B}) \cdot \underline{C} = \begin{vmatrix} 2 & 1 & 4 \\ 3 & 2 & p \\ 1 & 4 & 2 \end{vmatrix} = 42 - 7p = 0 \Rightarrow \underline{\underline{p = 6}}$$

Q(23). Find Curl ($\underline{F}$) and div($\underline{F}$) for the vector function $\underline{F}$= grad($x^2+y^2+z^2-3xyz$)

$\underline{F} = \nabla\phi = (2x - 3yz)\underline{i} + (2y - 3xz)\underline{j} + (2z - 3xy)\underline{k}$

$$\text{curl}\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2x - 2yz) & (2y - 3xz) & (2z - 3xy) \end{vmatrix} = 0$$

$$\text{div}\underline{F} = \nabla^2\phi = \frac{\partial}{\partial x}(2x - 3yz) + \frac{\partial}{\partial y}(2y - 3xz) + \frac{\partial}{\partial z}(2z - 3xy)$$

$$= 2 + 2 + 2 = 6$$

Q(24). If $\Phi = \dfrac{x}{r^3}$,

$$\frac{\partial\phi}{\partial x} = \frac{1}{r^3} - \frac{3x^2}{r^5}, \quad \frac{\partial\phi}{\partial y} = -\frac{3xy}{r^5}, \text{ and } \frac{\partial\phi}{\partial z} = -\frac{3xz}{r^5}$$

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{3x^2}{r^5} - \frac{6x}{r^5} + \frac{15x^3}{r^7}, \quad \frac{\partial^2\phi}{\partial y^2} = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}, \quad \frac{\partial^2\phi}{\partial z^2} = -\frac{3x^2}{r^5} + \frac{15xz^2}{r^7},$$

Therefore, $\nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0$

Q(25). If $\underline{r} = \underline{x}i + y\underline{i} + z\underline{k}$ then, $\text{div}(\underline{r}) = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3.$ and

$$\text{curl}\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

Use the results, $\text{div}(\phi\underline{A}) = \phi\text{div}(\underline{A}) + \nabla\phi \cdot \underline{A}$ and $\nabla f(r) = \dfrac{df}{dr}\dfrac{\underline{r}}{r}$

$$\text{div}(r^n r) = r^n \text{div}\underline{r} + \nabla(r^n) \cdot \underline{r}$$

$$= 3r^n + nr^{n-2}\,\underline{r} \cdot \underline{r}$$

$$= 3r^n + nr^n = \underline{\underline{(3+n)r^n}}$$

If $\Phi = \frac{1}{r}$,

$$\frac{\partial \phi}{\partial x} = -\frac{x}{r^3}, \quad \frac{\partial \phi}{\partial y} = -\frac{y}{r^3}, \text{ and } \frac{\partial \phi}{\partial z} = -\frac{z}{r^3}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{3x^2}{r^5} - \frac{1}{r^3}, \quad \frac{\partial^2 \phi}{\partial y^2} = \frac{3y^2}{r^5} - \frac{1}{r^3}, \quad \frac{\partial^2 \phi}{\partial z^2} = \frac{3z^2}{r^5} - \frac{1}{r^3}$$

Therefore, $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

Again use the results, $\text{curl}(\phi \underline{A}) = \phi \text{curl}(\underline{A}) + \nabla\phi \times \underline{A}$ and $\nabla f(r) = \frac{df}{dr}\frac{\underline{r}}{r}$

$$\begin{aligned} \text{curl}(r^n r) &= r^n \text{curl}\underline{r} + \nabla(r^n) \times \underline{r} \\ &= 0 + nr^{n-2}\underline{r} \times \underline{r} \\ &= 0 \end{aligned}$$

Q(26). Let $\underline{r} = \underline{x}i + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then,

$$\Rightarrow \underline{r} x \underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ a & b & c \end{vmatrix} = (yc - zb)\underline{i} + (az - cx)\underline{j} + (bx - ay)\underline{k}$$

Therefore, $\text{curl}(\underline{r} \times \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yc - zb & az - cx & bx - ay \end{vmatrix} = -2\underline{a}$

$$\Rightarrow \text{div}(\underline{r} x \underline{a}) = \frac{\partial}{\partial x}(yc - zb) + \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(bx - ay) = 0$$

$$\begin{aligned} \Rightarrow \underline{r} \cdot \underline{a} &= ax + by + cz \\ \nabla(\underline{r} \cdot \underline{a}) &= \nabla(ax + by + cz) = \underline{a} \end{aligned}$$

Q(27). Let $\underline{r} = \underline{x}i + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then, and $\underline{b} = d\underline{i} + e\underline{i} + f\underline{k}$

$$\Rightarrow \underline{r}x\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x & y & z \\ a & b & c \end{vmatrix} = (yc - zb)\underline{i} + (az - cx)\underline{j} + (bx - ay)\underline{k}$$

$$\Rightarrow (\underline{r}x\underline{a}) \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ yc - zb & az - cx & bx - ay \\ d & e & f \end{vmatrix}$$

$$= [f(az - cx) - e(bx - ay)]\underline{i} + [d(bx - ay) - f(yc - zb)]\underline{j} + [e((yc - zb) - d(az - cx)]\underline{k}$$

$$\Rightarrow \text{div}[(\underline{r}x\underline{a}) \times \underline{b}]$$

$$= \frac{\partial}{\partial x}[f(az - cx) - e(bx - ay)] + \frac{\partial}{\partial y}[d(bx - ay) - f(yc - zb)] + \frac{\partial}{\partial z}[e((yc - zb) - d(az - cx)]$$

$$\text{curl}[(\underline{r} \times \underline{a}) \times \underline{b}] = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fac - fcx - ebx + eay & dbx - day - fyc + fzb & eyc - ezb - daz + dcx \end{vmatrix} = 2\underline{b} \times \underline{a}$$

Q(28). (i) Let $\underline{r} = \underline{x}i + y\underline{i} + z\underline{k}$ and $\underline{a} = a\underline{i} + b\underline{i} + c\underline{k}$ then $(\underline{a} \circ \underline{r}) = ax + by + cz$

Therefore, $\frac{(\underline{a} \circ \underline{r})}{r^3}\underline{r} = \frac{ax + by + cz}{r^3}\underline{r}$

$$\text{curl}\frac{(\underline{a} \circ \underline{r})}{r^3}\underline{r} = \text{curl}\left(\frac{ax + by + cz}{r^3}\underline{r}\right) = \frac{ax + by + cz}{r^3}\text{curl}\underline{r} + \nabla\left(\frac{ax + by + cz}{r^3}\right) \times \underline{r}$$

$$= 0 + \nabla\left(\frac{ax + by + cz}{r^3}\right) \times \underline{r}$$

$$= \left\{(ax + by + cz)\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla(ax + by + cz)\right\} \times \underline{r}$$

$$= \left\{(ax + by + cz)\frac{(-\underline{r})}{r^5} + \frac{1}{r^3}\underline{a}\right\} \times \underline{r}$$

$$= \frac{\underline{a} \times \underline{r}}{r^3}$$

(ii)

$$\operatorname{div}\frac{(\underline{a}\circ\underline{r})}{r^3}\underline{r} = \operatorname{div}\left(\frac{ax+by+cz}{r^3}\underline{r}\right) = \frac{ax+by+cz}{r^3}\operatorname{div}\underline{r} + \nabla\left(\frac{ax+by+cz}{r^3}\right)\cdot\underline{r}$$

$$= \frac{ax+by+cz}{r^3}3 + \nabla\left(\frac{ax+by+cz}{r^3}\right)\cdot\underline{r}$$

$$= \frac{3(ax+by+cz)}{r^3} + \left\{(ax+by+cz)\nabla\left(\frac{1}{r^3}\right) + \frac{1}{r^3}\nabla(ax+by+cz)\right\}\cdot\underline{r}\ a$$

$$= \frac{3(ax+by+cz)}{r^3} + \left\{(ax+by+cz)\frac{(-3\underline{r})}{r^5} + \frac{1}{r^3}\underline{a}\right\}\cdot\underline{r}$$

$$= \frac{\underline{a}\cdot\underline{r}}{r^3}$$

(iii) $\Rightarrow\ \underline{a}x\underline{r} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a & b & c\\ x & y & z\end{vmatrix} = (bz-cy)\underline{i} + (cx-az)\underline{j} + (ay-bx)\underline{k}$

Use the results, $\operatorname{curl}(\phi\underline{A}) = \phi\operatorname{curl}(\underline{A}) + \nabla\phi\times\underline{A}$ and $\nabla f(r) = \frac{df}{dr}\frac{\underline{r}}{r}$

$$\operatorname{curl}(\underline{a}\times\underline{r}) = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -(yc-zb) & -(az-cx) & -(bx-ay)\end{vmatrix} = 2\underline{a}$$

$$\operatorname{Curl}\left(\frac{\underline{a}\times\underline{r}}{r^3}\right) = \frac{\operatorname{curl}(\underline{a}\times\underline{r})}{r^3} + \nabla\left(\frac{1}{r^3}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \nabla\left(\frac{1}{r^3}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \left(-\frac{3\underline{r}}{r^5}\right)\times(\underline{a}\times\underline{r})$$

$$= \frac{2\underline{a}}{r^3} + \left(-\frac{3}{r^5}\right)\underline{r}\times(\underline{a}\times\underline{r}) = \frac{2\underline{a}}{r^3} + \left(-\frac{3}{r^5}\right)[r^2\underline{a} - (\underline{r}\cdot\underline{a})\underline{r}$$

$$\operatorname{Curl}\left(\frac{\underline{a}\times\underline{r}}{r^3}\right) = -\frac{\underline{a}}{r^3} + \frac{3\underline{r}}{r^3}(\underline{a}\circ\underline{r})$$

Q(29). If $\underline{a}$ and $\underline{b}$ are constant vectors and α is a scalar quantity satisfy a vector equation $\alpha\underline{x} + \underline{a}\times\underline{x} = \underline{b}$, solve the vector equation for $\underline{x}$ for $\begin{cases}\alpha \neq 0 \\ \alpha = 0\end{cases}$

When $\alpha \neq 0$

$$\alpha\underline{x} + \underline{a}\times\underline{x} = \underline{b} \qquad (1)$$

$$\underline{a}\times(1) \;\Rightarrow\; \alpha\underline{a}\times\underline{x} + \underline{a}\times(\underline{a}\times\underline{x}) = \underline{a}\times\underline{b} \qquad (2)$$

$$\underline{a}\cdot(1) \;\Rightarrow\; \alpha\underline{a}\cdot\underline{x} + \underline{a}\cdot(\underline{a}\times\underline{x}) = \underline{a}\cdot\underline{b} \qquad (3)$$

From (1) $\Rightarrow\; \alpha\underline{a}\times\underline{x} + (\underline{a}\cdot\underline{x})\,\underline{a} - (\underline{a}\cdot\underline{a})\,\underline{x} = \underline{a}\times\underline{b}$ (4)

From (3) $\Rightarrow\; \alpha\underline{a}\cdot\underline{x} = \underline{a}\cdot\underline{b}$ and $\underline{a}\cdot\underline{x} = \dfrac{\underline{a}\cdot\underline{b}}{\alpha}$ (5)

|From (4) and (5) $\Rightarrow\; \alpha(\underline{b} - \alpha\underline{x}) + \dfrac{(\underline{a}\cdot\underline{b})}{\alpha}\underline{a} - (\underline{a}\cdot\underline{a})\,\underline{x} = \underline{a}\times\underline{b}$

$$\Rightarrow\; \underline{x} = \frac{\underline{a}\times\underline{b} - \dfrac{(\underline{a}\cdot\underline{b})}{\alpha}\underline{a} - \alpha\underline{b}}{\alpha^2 + a^2}$$

When $\alpha = 0$

$$\underline{a}\times\underline{x} = \underline{b} \qquad (1)$$

$$\underline{a}\times(1) \;\Rightarrow\; \underline{a}\times(\underline{a}\times\underline{x}) = \underline{a}\times\underline{b} \qquad (2)$$

From (1) $\Rightarrow\; (\underline{a}\cdot\underline{x})\,\underline{a} - (\underline{a}\cdot\underline{a})\,\underline{x} = \underline{a}\times\underline{b}$ (3)

Let $\Rightarrow\; (\underline{a}\cdot\underline{x}) = t$(parameter)

|From (3) $\Rightarrow\; \underline{x} = \dfrac{t\,\underline{a} - \underline{a}\times\underline{b}}{a^2}$

31. Find the unit tangent vector to the surface of the parboiled of revolution $z = x^2 + y^2$ at the point (1,2 ,5)

32. Find the equation of tangent plane and normal line to the surface $z = x^2 + y^2$ at the point (2,-1,5)

33. Evaluate $\nabla \circ \left(r\nabla(\frac{1}{r^3}) \right)$ and Evaluate $\nabla^2 \left[\nabla \circ (\frac{\underline{r}}{r^2}) \right]$

If f(r) is a function of r only then find f(r) such that $\nabla^2 f(r) = 0$

34. Prove that Curl (Curl($\underline{A}$))$= -\nabla^2 \underline{A} + \nabla(\nabla \circ \underline{A})$

If $\underline{v} = \underline{\omega} \times \underline{r}$ prove that $\underline{\omega} = \frac{1}{2}\text{curl}(\underline{v})$ where $\underline{\omega}$ is a constant vector.

(5, 2, 0), (2, 1, 3) and (4, 1, -1)