Simplex Method

The graphical method shows that the optimum LP solution is always associated with a corner point of the solution space. This result is key to the development of the general algebraic simplex method for solving any LP model.

The computational procedure of the simplex method determines the corner points algebraically, by first converting all the inequality constraints into equations and then manipulating the resulting equations in a systematic manner.

A main feature of the simplex method is that it solves the LP iterations. Each iteration moves the solution to a new corner point that has the potential to improve the value of the objective function. The process ends when no further improvements can be realized.

Preparation for the Simplex method

Standard form of the linear programming problem

In preparation for the use of the Simplex method, it is necessary to express the linear programming problem in **standard form**. For a linear program with n variables and m constraints, we will use the following standard form:

maximize $z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2v}x_n &= b_2 \\ &. \\ &. \\ &. \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where the variables $x_1,..., x_n$ are non-negative, and the constants b_l, ..., b_m on the right-hand sides of the constraints are also non-negative. We can use matrix notation to represent the cost (or profit) vector c = (c_l, c_2,..., c_n) and the decision variable vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

The coefficient matrix is

$$A = \begin{bmatrix} a_{11} \dots a_{1n} \\ . \\ . \\ . \\ a_{m1} \dots a_{mn} \end{bmatrix}$$

And the requirement vector is $b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$

Then the optimization problem can be expressed as

maximize $z = cx$

subject to $Ax = b$

$$x \geq 0$$
$$b \geq 0$$

To convert a minimization problem to a maximization problem, we can simply multiply the objective function by -1, and then maximize this function. (Recall that there are no sign restrictions on the c_i). For example, the problem of minimizing $z = 3x_1 - 5x_2$ is equivalent to maximizing $z = -3x_1 + 5x_2$. Negative right-hand sides of the constraints can be made positive by multiplying the constraint by -1 (reversing the sense of the inequality).

Equality constraints require no modification. Inequality constraints can be converted to equalities.

Converting Inequalities into Equations

In ($\leq$) constraints, the right-hand side can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and the left-hand side of the ($\leq$) constraint thus yields the unused or slack amount of the resource.

To convert a ($\leq$) inequality to an equation, a nonnegative slack variable is added to the left-hand side of the constraint. For example1, the constraint associated with the use of raw material M1 is given as

$$6x_1 + 4x_2 \leq 24$$

Defining s_1 as the slack or unused amount of Ml, the constraint can be converted to

the following equation

$$6x_1 + 4x_2 + s_1 = 24, s_1 \geq 0$$

Next, a ($\geq$) constraint normally sets a lower limit on the activities of the LP model. As such, the amount by which the left-hand side exceeds the minimum limit represents a surplus.

The conversion from ($\geq$) to (=) is achieved by subtracting a nonnegative surplus variable from the left-hand side of the inequality. For example, in the example3 the constraint representing the minimum feed requirements is given as

$$x_1 + x_2 \geq 800$$

Defining s_1 as the surplus variable, the constraint can be converted to the following equation.

$$x_1 + x_2 - s_1 = 800, s_1 \geq 0$$

Note importantly that the slack and surplus variables, s_1 and S_1> are always nonnegative.

the slack and surplus variables are going to be treated exactly like any other decision variable throughout the solution process. In fact, their final values in the solution of the linear programming problem may be just as interesting to a systems manager or analyst as are the values of the original decision variables.

Finally, all variables are required to be non-negative in the standard form. The reason for placing problems in standard form is that our general solution method will be seen to operate by finding and examining solutions to the system of linear equations Ax = b (ie., by finding values of the decision variables that are consistent with the problem constraints), with the aim of selecting a solution that is optimal with respect to the objective function.

Simplex method can be summarized succinctly as follows

Step 1: Examine the elements in the top row (the objective function row). If all elements are ≥ 0, .then the current solution is optimal; stop. Otherwise go to Step 2.

Step 2: Select as the non-basic variable to enter the basis that variable corresponding to the most negative coefficient in the top row. This identifies the pivot column.

Step 3: Examine the coefficients in the pivot column. If all elements are ≤ 0 ,then this problem has an unbounded solution (no optimal solution); stop. Otherwise go to Step 4.

.Step 4: Calculate the ratios $r_i = {b_i}/{a_{ik}} \quad for all i = 1,2,\ldots,m \; for which a_{ik} > 0$

where a_{ik} is the i-th element in the pivot column k. Then select

$$r = \min\{r_i\}$$

This identifies the **pivot row** and defines the variable that will leave the basis. The **pivot element** is the element in the pivot row and pivot column.

Step 5: To obtain the next tableau (which will represent the new basic feasible solution), divide each element in the pivot row by the pivot element. Use this row now to perform row operations on the other rows in order to obtain zeros in the rest of the pivot column, including the z row. This constitutes a pivot operation, performed on the pivot element, for the purpose of creating a unit vector in the pivot column, with a coefficient of one for the variable chosen to enter the basis

Lets apply these steps to the initial tableau in our example problems.