3. Partial Differential Equations

Classification: The most general linear partial differential equation of the second order with two independent variables is,

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+F\phi+G=0$$

where A, B, C, D, E, F, G are functions of x and y including constants.

(i) If $B^2<4AC$ **the equation is said to be elliptic.**

(ii) If $B^2>4AC$ **the equation is hyperbolic.**

(iii) If $B^2=4AC$ **the equation is said to be parabolic.**

Examples: (i) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$ which is known as Laplace equation is elliptic.

(ii) $c^2\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\phi}{\partial t^2}$ which is known as one dimensional wave equation is hyperbolic.(c is a constant)

(iii) $k^2\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial\varphi}{\partial t}$ which is known as the diffusion equation is parabolic.(k is a constant)

The order of a partial differential equation is the order of the highest partial derivative it contains. A partial differential equation is said to be linear if all partial derivatives and the dependent variable occur as first order terms.

In general , a partial differential equation of order n has a solution which contains at most n arbitrary functions. Therefore the general solution can be written as the linear combination of n arbitrary functions. This general solution can be particularized to a unique solution if appropriate extra conditions are provided. These are classified as boundary conditions. The kind of boundary conditions we need to specify depend on the nature of the problem.

Classification of problems

(i) Equilibrium problems
(ii) Propagation problems

Equilibrium problems relate to steady state conditions. The problems in this category include steady state temperature distributions, steady flow of electric current, equilibrium stress situations steady ideal fluid flows. These problems are boundary value problems. We need to specify conditions which exist along the entire boundary.

Examples

(i) Laplace equation in two dimensional Cartesian coordinates.

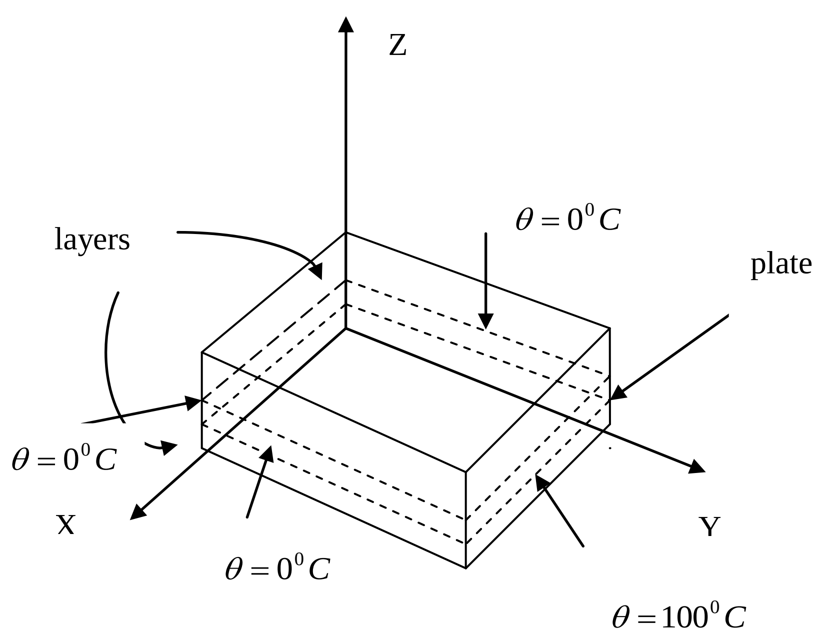
i.e. $$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$$

(ii) Poisson equation

i.e. $$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=f(x,y)$$

Two Dimensional Heat Flow

Suppose we want to find the temperature distribution in a rectangular metal plate under certain conditions. The plate is covered on its top and bottom faces by layers of thermal insulating material so that heat is constrained to flow mainly in the X and Y directions. Along the edges of the plate various conditions are applied. These are known as boundary conditions.



When formulating a simple mathematical model the following assumptions are made.

(i) The metal is uniform in the sense that its thermal conductivity is the same at all points of the plate.
(ii)The plate is sufficiently thin so that we neglect any heat flow in the directions perpendicular to its face.

(iii) The temperature distribution is in the steady state. i.e. temperature at any point in the plate does not depend on the time.

Let temperature function depends on x and y. Therefore,

$$\theta \equiv \theta(x, y)$$

It can be shown that $\theta(x, y)$ satisfies Laplace equation $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$.

Laplace equation possesses infinitely many solutions. For a unique solution the following boundary conditions are used.

Boundary conditions

$\theta = 0$ **when** $x = 0$ **for** $0 \le y < b$ **…………..(i)**

$\theta = 0$ **when** $y = 0$ **for** $0 < x < a$ **………….(ii)**

$\theta = 0$ **when** $x = a$ **for** $0 < y < b$ **………..(iii)**

$\theta = 100$ **when** $y = b$ **for** $0 < x < a$ **……….(iv)**

The analytical solution i.e. A formular for θ is obtained by the method of separation of variables.

Assume that we can express $\theta(x, y)$ as a product of a function of x and a function of y .

$$\theta = X(x).Y(y)$$
$$\frac{\partial \theta}{\partial x} = X'(x)Y(y) \Rightarrow \frac{\partial^2 \theta}{\partial x^2} = X''(x)Y(y)$$
$$\frac{\partial \theta}{\partial y} = X(x)Y'(y) \Rightarrow \frac{\partial^2 \theta}{\partial y^2} = Y''(y)X(x)$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \Rightarrow YX'' + XY'' = 0$$

$X''Y = -XY''$

$\frac{X''}{X} = -\frac{Y''}{Y} = \pm k^2, k \neq 0$

(If k=0 we get $\frac{X''}{X} = 0 \Rightarrow X(x) = Ax + B$

$\theta(x, y) = (Ax + B)Y(y)$

When x=0, $\theta = 0 \Rightarrow 0 = B(Y(y)) \Rightarrow B = 0.$

When x=a, $\theta = 0 \Rightarrow 0 = (Aa)Y(y) \Rightarrow A = 0$

X=0 . This is not possible for non trivial solutions.)

If $\frac{X''}{X} = -\frac{Y''}{Y} = k^2$

$X'' = k^2 X \Rightarrow \frac{d^2X}{dx^2} - k^2 X = 0 \Rightarrow X = Ae^{kx} + Be^{-kx}$

$Y'' = k^2 Y \Rightarrow \frac{d^2Y}{dy^2} + k \ \ y = 0 \Rightarrow Y = CCosky + DSinky$

$\theta(x, y) = (Ae^{kx} + Be^{-kx})(Cosky + DSinky)$ ———————— (1)

If $\frac{X''}{X} = \frac{Y''}{Y} = -k^2$

$X'' = -k^2 X \Rightarrow X = A'Coskx + B'Sinkx$

$Y'' = k^2 Y \Rightarrow Y = C'e^{ky} + D'e^{-ky}$

$\theta(x, y) = (A'Coskx + B'Sinkx)(C'e^{ky} + D'e^{-ky})$ ———————— (2)

(1) $\Rightarrow$ when $x = 0, \theta = 0$

0=(A+B)Y(y) ,for 0<y<b $\Rightarrow$ B=-A (for non trivial θ)

Then $\theta(x, y) = (Ae^{kx} - Ae^{-kx})Y(y)$

$\theta(x, y) = A''(Sinhkx)Y(y)$

$\theta = 0$ when x=a.

0= $A''(Sinhkx)Y(y) \Rightarrow Sinhka = 0$ (Since $A'' \neq 0, Y(y) \neq 0$ for non trivial θ.)

ka=0 . This is a contradiction since $k \neq 0, a \neq 0.$

(1) is not possible.

The possible solution is (2).

(2) $\Rightarrow$ $\theta(x, y) = (A'Coskx + B'Sinkx)(C'e^{ky} + D'e^{-ky})$

When $x = 0, \theta = 0$ for $0 < y < b$

$$0 = A'(Y(y)) \Rightarrow A' = 0$$

$$\theta(x,y) = (B'Sinkx)(C'e^{ky} + D'e^{-ky})$$

When y=0, $\theta = 0$ for $0 < x < a$

$0 = B'(Sinkx)(C' + D') \Rightarrow C' + D' = 0$ for non trivial solutions.

$$D' = -C'$$

$$\theta(x,y) = (C'B'Sinkx)(e^{ky} - e^{-ky})$$

$$\theta(x,y) = (E'Sinkx)(Sinhky)$$

When $x = a, \theta(x,y) = 0$ for $0 < y < b$

$0 = E'SinkaSinhky \Rightarrow Sinka = 0$ for a non trivial solution.

Sin ka=0 $\Rightarrow ka = n\pi$ where n is an integer.

Then k= $\frac{n\pi}{a}$ $\therefore \theta = E'Sin\frac{n\pi x}{a}Sin(\frac{n\pi}{a})y$, , $n = \pm 1, \pm 2, ...$

When n is negative , let $n = -n'$

$$\theta = E'Sin(-\frac{n'\pi x}{a})Sinh(-\frac{n'\pi x}{a}) = E'Sin(\frac{n'\pi x}{a})Sinh(\frac{n'\pi x}{a})$$

Therefore, negative values for n can be neglected.

General Solution is $\theta = \sum_{n=1}^{\infty} A_n Sin(\frac{n\pi x}{a})Sinh(\frac{n\pi y}{a})$

$\theta = 100$ when y=b for $0 \le x \le a$

$\therefore$ $100 = \sum_{n=1}^{\infty} A_n Sin\frac{n\pi x}{a}Sinh\frac{n\pi b}{a}$

This represents fourier half –range Sine series for the function $f(x) = 100$.

$$A_n Sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a 100Sin\frac{n\pi x}{a}dx = \frac{200}{n\pi}(1 - Cosn\pi)$$

Hence $A_n Sinh\frac{n\pi b}{a} = \begin{cases} 0 & ,evenn \\ \frac{400}{n\pi} & ,oddn \end{cases}$

$$A_n = \begin{cases} 0 & n-even \\ \frac{400}{n\pi}\cdot\frac{1}{Sinh(\frac{n\pi b}{a})} & n-odd \end{cases}$$

General Solution is $\theta = \sum_{n=1}^{\infty} \frac{400}{n\pi}\frac{1}{Sinh(\frac{n\pi b}{a})}Sin(\frac{n\pi x}{a})Sinh(\frac{n\pi y}{a})$

Analytical method of the General Solution is made to yield qualitative and quantitative information. It will be seen later, even for a geometrically simple region straightforward boundary conditions and the resulting formula for θ is complicated.

Then if the geometry of the region is more awkward the analytical method may become vertually impossible. In such a case we can resort to a numerical method.

<u>Exercise</u>: A rectangular plate with insulated surfaces is 8cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge y=0 is given by, U(x,0)= 100Sin($\frac{\pi x}{8}$), $0 < x < 8,$ while the two long edges x=0 and x=8, as well as the other short edge are kept at $0^0 C$, find steady state temperature U(x,y).

<u>Laplace equation in polar coordinates</u>

It can be shown that Laplace equation in polar co-ordinates is

$$r^2 \frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0 .$$

<u>Example</u> Solve the above equation by the method of separation of variables.

Let $T(r,\theta) = R(r)\theta(\theta)$

$$\frac{\partial T}{\partial \theta} = R'\theta \Rightarrow \frac{\partial^2 T}{\partial r^2} = R''\theta$$

$$\frac{\partial T}{\partial \theta} = R\theta' \Rightarrow \frac{\partial^2 T}{\partial \theta^2} = R\theta''$$

$$r^2 \frac{\partial^2 T}{\partial r^2} + r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial \theta^2} = 0 \Rightarrow r^2 R''\theta + rR'\theta + R\theta'' = 0$$

$$(r^2 R'' + rR')\theta + R\theta'' = 0 \Rightarrow \quad \frac{r^2 R'' + rR'}{R} = \frac{-\theta''}{\theta} = h$$ **(say)**

<u>**If h>o ,**</u> **let** $h = k^2, k \neq 0$

$$r^2 R'' + rR' - k^2 R = 0$$

Let $r = e^z \Rightarrow z = \ln r \Rightarrow \dfrac{dr}{dz} = e^z = r$

$$\frac{dR}{dr} = \frac{dR}{dz} . \frac{dz}{dr} = \frac{dR}{dz} . \frac{1}{r} \Rightarrow r\frac{dR}{dr} = \frac{dR}{dz}$$

$$\Rightarrow \frac{d}{dr}\left(r\frac{dR}{dr}\right) = \frac{d}{dr}\left(\frac{dR}{dz}\right) = \frac{d}{dz}\left(\frac{dR}{dz}\right)\frac{dz}{dr} = \frac{d^2R}{dz^2} . \frac{1}{r}$$

$$\Rightarrow r\frac{d^2R}{dr^2} + \frac{dR}{dr} = \frac{d^2R}{dz^2} . \frac{1}{r}$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} = \frac{d^2R}{dz^2}$$

Let $D = \dfrac{d}{dz}$ **then** $r^2 R'' + rR' - k^2 R = 0$

$$\Longrightarrow \quad (D^2 - k^2)R = 0$$

Let $R = e^{\lambda z} \Rightarrow$ **auxiliary equation is**

$$\lambda^2 - k^2 = 0 \Rightarrow \lambda = \pm k$$

$$\therefore R = Ae^{kz} + Be^{-kz} = Ar^{k} + Br^{-k}$$

$$\theta'' = -k^2\theta \Rightarrow \theta = CCosk\theta + DSink\theta$$

$$T(r,\theta) = (\ Ar^{k} + Br^{-k})(CCosk\theta + DSink\theta)$$

…………(1)

<u>If h=0</u> $\theta'' = 0 \Rightarrow \theta = C'\vartheta + D'$

$$\Rightarrow D^2R = 0 \quad R(r) = A'r + B'$$

$$T(\ r,\vartheta) = (A'r + B')(\ C'\vartheta + D'\ )$$ …………….(2)

<u>If h<o</u> **let** $h = -k^2, k \neq 0.$

$$\theta'' = k^2\theta \Rightarrow \theta = C''e^{k\theta} + D''e^{=k\theta}$$

$$r^2R'' + rR' + k^2R = 0 \Rightarrow (D^2 + k^2)R = 0$$

$$R = e^{\lambda z} \Rightarrow \lambda^2 + k^2 = 0 \Rightarrow \lambda = \pm ki$$

$$R = (A''e^{kiz} + Be^{-kiz})(\ C''e^{k\theta} + D''e^{=k\theta})$$

$$R = (A'''Coskz + B'''Sinkz)(\ C''e^{k\theta} + D''e^{=k\theta})$$

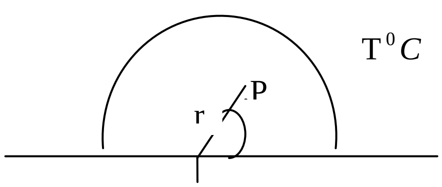
$$R = (A'''Cos(k\ln r) + B'''Sin(k\ln r)($$

$$C''e^{k\theta} + D''e^{=k\theta})$$ ………..(3)

There are three possible solutions given by (1),(2) and (3).

<u>Example</u>

The diameter of a semi circular plate of radius a, is kept at 0^0C
and the temperature at the semi circular boundary is T^0C.
Find the steady state temperature in the plate.



LET $u(r,\theta)$ **be the steady state temperature at any point P satisfies the equation,**

$$r^2\frac{\partial^2 T}{\partial r^2}+r\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial \theta^2}=0$$

The boundary conditions are,

(i) $U(r,\theta)=0,\quad 0\le r\le a$

(ii) $U(r,\pi)=0,\ 0\le r\le a$

(iii) $U(a,\theta)=T$, $0<\theta<\pi$

For conditions (i) and (ii) $U=0$ **when r=0 and** $U=0$ **when r=a.**

It can be shown that possible solution is ,

$$U(r,\theta)=(Ar^{k}+Br^{-k})(CCosk\theta+DSink\theta)$$

$U(r,0)=0\Rightarrow 0=R(r)C\Rightarrow C=0$ **(Since** $R(r)\neq 0)$

$$\therefore U(r,\theta)=(Ar^{k}+Br^{-k})(DSink\theta)$$

$$U(r,\pi)=0\Rightarrow 0=R(r)DSink\pi\Rightarrow Sink\pi=0$$

$$\therefore k\pi=n\pi,n=\pm1,\pm2,.......$$

$$k=n\ ,n=\pm1,\pm2,.......$$

$$U(r,\theta)=(Ar^{n}+Br^{-n})(DSinn\theta)$$

$U=0$ when $r=0, then B=0$.

Then $U(r,\theta)=\sum_{n=0}^{\infty}A_n r^n Sinn\theta$

When r=a , U =T,

$$\therefore T=\sum_{n=0}^{\infty}A_n a^n Sinn\theta, 0<\theta<\pi.$$

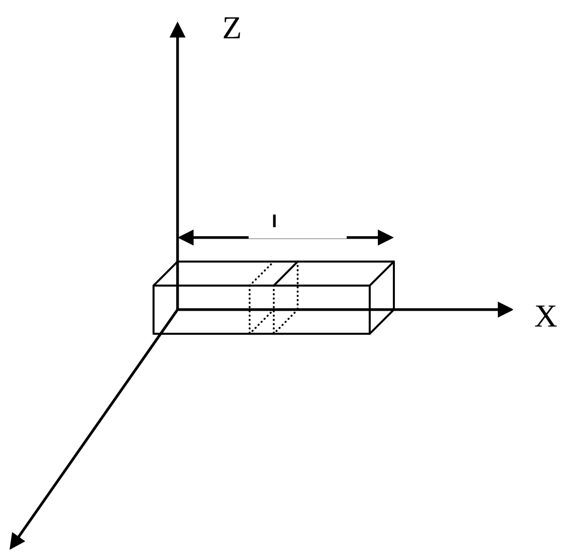
This is the fourier half range Sine series of T at all points.

$$A_n a^n=\frac{2}{\pi}\int_0^T TSinn\theta d\theta=\frac{2T}{n\pi}\left(1-(-1)^n\right)=\begin{cases}\dfrac{4T}{n\pi} & n-even\\ 0 & n-0dd\end{cases}$$

$$U(r,\theta) = \frac{4T}{\pi} \sum_{n=1,3,5...}^{\infty} \frac{1}{a^n n} r^n Sinn\theta$$

<u>Exercise</u>: An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π. This end is maintained at a temperature U_0 at all points and other ends are at a zero temperature. Determine the temperature at any point of the plate.

<u>Heat Flow in one dimension</u>



Suppose that we have a long thin bar of length l which is aligned along the x-axis . We wish to determine the temperature distribution $\theta(x,t)$ in the bar.

Assume the bar is insulated along its sides , and that the heat flows in the X direction only.
The following laws of heat flow are used.

(1) **The amount of heat in a body is proportional to its mass and to its temperature.**

(2) **Tha heat flows from a point at a higher temperature at a lower temperature.**

(3) **The rate of flow of heat through a plane surface is proportional to the area of the surface and to the rate of change of temperature with respect to the distance in a direction perpendicular to the plane.**

It can be shown that the temperature distribution $\theta(x,t)$ **satisfies the following.**

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{k}\frac{\partial\theta}{\partial t},$$ **where k is a positive constant.**

<u>**Example:**</u> **Obtain the solution of the equation** $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial\theta}{\partial t}$**, where k is a positive constant satisfying the boundary conditions,**

$U(0,t)=0, U(l,t)=0,$ **t** ≥ 0 **, U(x,0)=f(x),** $0 < x < l$. **f(x) is a given function and l is a constant.**

Let $U(x,t) \; = X(x).T(t)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \Rightarrow X''(x)T''(t) = \frac{1}{k}X(x).T'(t)$$

$$\frac{X''}{X} = \frac{T'}{kT} = \lambda($$ **since x and t are independent)**

f^n **of x** f^n **of t**

Since the temperature of a point on the bar is decreasing when t increases, λ **is negative.**

Let λ **= -** ω^2 , $\omega \neq 0$

$$\frac{1}{X} \cdot \frac{d^2X}{dx^2} = -\omega^2 \Rightarrow X(x) = ACos\,\omega x + BSin\,\omega x$$ **, A,B are constants.**

$$\int \frac{T'}{T} dt = \int -\omega^2 k dt \Rightarrow \ln T = -\omega^2 kt \Rightarrow T(t) = Ce^{-\omega^2 kt}$$ **, where C is a constant.**

$$U(x,t) = (ACos\,\omega x + BSin\,\omega x)\, Ce^{-\omega^2 kt}$$

$$= (A'Cos\,\omega x + B'Sin\,\omega x) e^{-\omega^2 kt} \qquad \textbf{(1)}$$

$U(0,t) = 0, \forall t \geq 0$, $(1) \Rightarrow 0 = A'e^{-\omega^2 kt}$, $\forall t \geq 0 \Rightarrow A' = 0$

$$U(x,t) = B'Sin\,\omega x . e^{-\omega^2 kt}$$

$U(l,t) = 0, \forall t \geq 0 \Rightarrow B' = 0$ **or** $Sin\,\omega l = 0$

B' **is not possible for a non trivial solution.**

$$\therefore Sin\,\omega l = 0 \Rightarrow \omega l = r\pi$$

(r is an integer)

$$\omega = \frac{r\pi}{l}, r = \pm 1, \pm 2, \ldots\ldots\ldots\ldots$$

$$U(x,t) = B' Sin(\frac{r\pi x}{l}) e^{-\frac{r^2\pi^2 kt}{l^2}} \quad , \quad r = \pm 1, \pm 2, \ldots\ldots\ldots$$

$$U(x,t) = \sum_{r=1}^{\infty} B_r Sin \frac{r\pi x}{l} e^{-\frac{r^2\pi^2 kt}{l^2}}$$ **where** B_r **is a constant depending on r.**

$$U(x,0) = f(x), 0 < x < l \Rightarrow f(x) = \sum_{r=0}^{\infty} B_r Sin \frac{r\pi x}{l}, o < x < l$$

This represents Fourier half range Sine series of f(x).

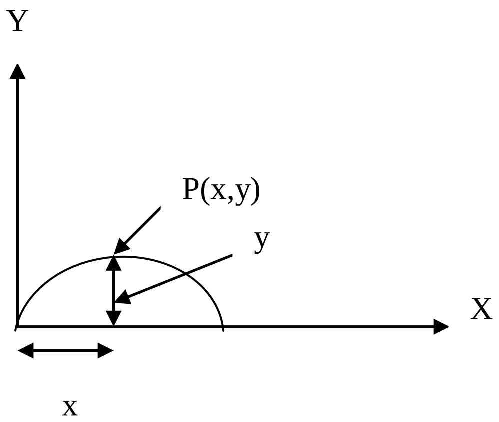
$$\because B_r = \frac{2}{l} \int_0^l f(x) Sin\left(\frac{r\pi x}{l}\right) dx, r = 1,2,3\ldots..$$

$$U(x,t) = \sum_{r=1}^{\infty} \left(\frac{2}{l} \int_0^l f(x) Sin\left(\frac{r\pi x}{l}\right) dx \right) Sin \frac{r\pi x}{l} e^{-\frac{r^2\pi^2 kt}{l^2}}$$

<u>Example</u>:

A rod of length l has its A and B kept at $0^0 C$ **and** $100^0 C$ **respectively, until steady state conditions prevail. If the temperature at B is reduced suddenly to** $0^0 C$ **and kept so , while that of A is maintained . Find the temperature** $U(x,t)$ **at a distance x from A and at a distance x.**

Equation of Vibrating String(One Dimensional Wave Equation)



Consider a perfectly flexible homogeneous string tightly stretched between two points O and A. We assume the tension in the string to be so large that gravity may be neglected in comparison with it.

The differential equation governing the motion when the string is set vibrating in the vertical plane can be shown as,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$ **where c is a constant.**

This is known as the one dimensional wave equation.

Example:

A string is stretched and fastened to two points at a distance l apart . Motion is started by displacing the string in the form $y = aSin\frac{\pi x}{l}$ **from which it is released at a time t=o. Show that the displacement of any point at distance x from one end at time t is given by,**

$$y(x,t) = aSin\frac{\pi kt}{l}Cos\frac{\pi kt}{l}$$ **where k is a constant.**

Let $y(x,t) = X(x)T(t)$

$y(x,t)$ **satisfies the equation** $\frac{\partial^2 y}{\partial t^2} = k^2\frac{\partial^2 y}{\partial x^2} \Rightarrow XT'' = k^2 X''T \Rightarrow \frac{X''}{X} = \frac{1}{k^2}\frac{T''}{T} = h$

When $h > 0$ **i.e.** $h = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$
$$T'' = \lambda^2 k^2 T \Rightarrow T(t) = C_3 e^{\lambda kt} + C_4 e^{-\lambda kt}$$

$$Y(x,t) = X(x).T(t) = (C_1 e^{\lambda x} + C_2 e^{-\lambda x})(C_3 e^{\lambda kt} + C_4 e^{-\lambda kt})$$ **----------------(1)**

<u>When $h = 0$</u>

$$X'' = 0 \Rightarrow X(x) = C_5 x + C_6$$
$$T'' = 0 \Rightarrow T(t) = C_7 t + C_8$$
$$Y(x,t) = X(x).T(t) = (C_5 x + C_6)\ (C_7 t + C_8)$$ **----------------(2)**

<u>When $h < 0$</u> **i.e. $h = -\lambda^2$**

$$X'' = -\lambda^2 X \Rightarrow X(x) = C_9 Cos\lambda x + C_{10} Sin\lambda x$$
$$T'' = -\lambda^2 k^2 T \Rightarrow T(t) = C_{11} Cos\pi kt + C_{12} Sin\lambda kt$$
$$Y(x,t) = X(x).T(t) = (C_9 Cos\lambda x + C_{10} Sin\lambda x)(C_{11} Cos\pi kt + C_{12} Sin\lambda kt)$$ **---------------(3)**

Since this is a problem on vibrating string , the solution should be a periodic function of t. Therefore, the solution is given by (3).

<u>Boundary conditions are</u>:

(i) $y(0,t) = y(l,t) = 0, \forall t$

(ii) $y(x,0) = aSin\dfrac{\pi x}{l}$

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$

(3) $\Rightarrow y(0,t) = C_9(T(t)) = 0 \Rightarrow C_9 = 0$ **for non-trivial solutions.**

$\therefore y(x,t) = C_{10} Sin\lambda x(C_{11} Cos\pi kt + C_{12} Sin\lambda kt)$

$y(l,t) = 0 \Rightarrow C_{10} Sin\lambda l(T(t)) = 0 \Rightarrow Sin\lambda l = 0$ **for non-trivial solutions.**

$\lambda l = n\pi$ **, n is an integer**

$\lambda = \dfrac{n\pi}{l}, n = \pm 1, \pm 2, ...$

$\therefore y(x,t) = Sin\dfrac{n\pi x}{l}\left(b_1 Cos\dfrac{n\pi kt}{l} + b_2 Sin\dfrac{n\pi kt}{l}\right)$**,** b_1, b_2 **are non zero real numbers.**

$$\frac{\partial y}{\partial t} = Sin\frac{n\pi x}{l}\left(-b_1\left(\frac{n\pi k}{l}\right)Sin\frac{n\pi kt}{l} + b_2\left(\frac{n\pi k}{l}\right)Cos\frac{n\pi kt}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \Rightarrow b_2 = 0$$

$$\therefore y(x,t) = b_1 Sin\frac{n\pi x}{l} Cos\frac{n\pi kt}{l}, n = \pm 1, \pm 2, ...$$

General solution is $\therefore y(x,t)=\sum_{n=1}^{\infty} A_n Sin\frac{n\pi x}{l} Cos\frac{n\pi kt}{l}$ **where** A_n **is a constant depending on n.**

$y(x,0)=aSin\frac{\pi x}{l} \Rightarrow \quad aSin\frac{\pi x}{l}=\sum_{n=1}^{\infty} A_n Sin\frac{n\pi x}{l}$

$A_1=a, A_i=0$ **for all** $i \geq 2$ **, where i is an integer.**

Solution is $\therefore y(x,t)=aSin\frac{\pi x}{l} Cos\frac{\pi kt}{l}$ **.**

Example:

A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y=c(lx-x^2)$ **from which it is released at time t=0. Find the displacement at any point on the string at a distance x from one end at time t.**

Exercises

1. **Solve the following equation.**

   $\frac{\partial^2 u}{\partial x^2}=24x^2(t-2)$ **given that at** $x=0, u=e^{2t}$ **and** $\frac{\partial u}{\partial x}=4t$ **.**

2. **A rectangular plate OPQR is bounded by the lines x=0,y=0, x=4, y=2. Determine the potential distribution u(x,y) over the rectangle using the Laplace equation** $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$**, subject to the following boundary conditions**

   $u(0,y)=0$ **,** $0 \leq y \leq 2$
   $u(0,y)=0$ **,** $0 \leq y \leq 2$
   $u(x,2)=0$ **,** $0 \leq x \leq 4$
   $u(x,0)=x(4-x)$ **,** $0 \leq x \leq 4$**.**

3. **Solve Laplace's equation in plane polar coordinates**

   $\frac{\partial^2 v(r,\theta)}{\partial r^2}+\frac{1}{r}\frac{\partial v(r,\theta)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v(r,\theta)}{\partial \theta^2}=0$ **in the circular region** $x^2+y^2=1$ **of the plane where,**

   (1) $v(r,\theta)$ **is finite for** $0 \leq x \leq 1$ **and for all** θ
   (2) $v(1,\theta)=Sin2\theta-4Cos\theta$
   (3) $v(r,\theta+2\pi)=v(r,\theta)$ **for** $0 \leq x \leq 1$**.**

4. **An insulated uniform metal bar , 10 units long , has the temperature of its ends maintained at** 0^0C **and at t=0 the temperature distribution** $f(x)$ **along the bar is defined by** $f(x)=x(10-x)$**. Solve the heat conduction equation** $\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial u}{\partial t}$ **with** $c^2=4$ **to determine the temperature u of any point in the bar at time t.**

5. A perfectly elastic string is stretched between two points 10cm apart. Its centre point is displaced 2cm from its position of rest at right angles to the original direction of the string and then released with zero velocity. Applying the equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ **with** $c^2 = 1$, **determine the subsequent motion** $u(x,t)$.