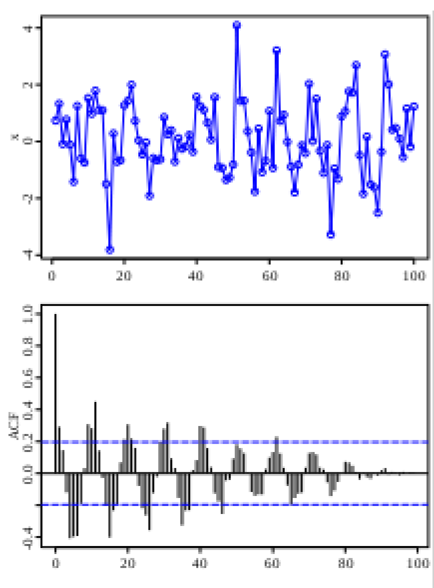
Stationary Processes

- **Strictly stationary**
  - The whole joint distribution is independent of the date at which it is measured and depends only on the lag.
  - $E(y_t)$ is a finite constant.
  - $V(y_t)$ is a finite constant.
  - $Cov(y_t, y_{t-s})$ depends only on the lag s (independent of time)
- **Weakly stationary(Second order stationary)**
  - $E(y_t)$ is a finite constant.
  - $Cov(y_t, y_{t-s})$ depends only on the lag s (independent of time)

Correlogram

- This is a plot of the sample autocorrelations r_k versus k (the time lags).
- The correlogram is a commonly-used tool for checking randomness in a data set. If random, autocorrelations should be near zero for any and all time-lag separations. If non-random, then one or more of the autocorrelations will be significantly non-zero.
- If a time series has tendency to alternate with successive observations on different sides of the overall mean then correlogram also tense to alternate.
- If a time series contains seasonal fluctuations , then the correlogram also exhibit oscillations of the same frequency.
- If a time series contains a trend then the values of r_k will come down to zero, except for the large values of lag.



The two figures show a time plot of 100 random numbers (above) and the **correlogram** of the series.(below).

Note: It is advisable to remove trend,seasonal fluctuations and outliers before calculating correlation coefficients, as they hide the other important features of time series.

Unit Root Tests (Tests for Stationarity)

Dickey Fuller (DF)Test is significant $\Rightarrow$ time series is non Stationary

Augmented Dicky Fuller (ADF)Test is significant $\Rightarrow$ time series is Stationary

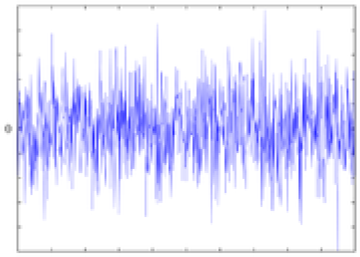
General Properties of Autocorrelation Function

- $\rho(\tau) = \rho(-\tau)$
- $|\rho(\tau)| \leq 1$
- Lack of uniqueness

Basic Properties of time series

- White Noise Process
  - $y_t = \varepsilon_t \, , \, \varepsilon_t \approx i.i.d(0, \sigma^2)$
  - Simple possible stationary random sequence
- A time series may have both stochastic and deterministic Components :

$$y_t = \alpha + \beta t + \varepsilon_t \, , \, \varepsilon_t \approx i.i.d(0, \sigma^2)$$



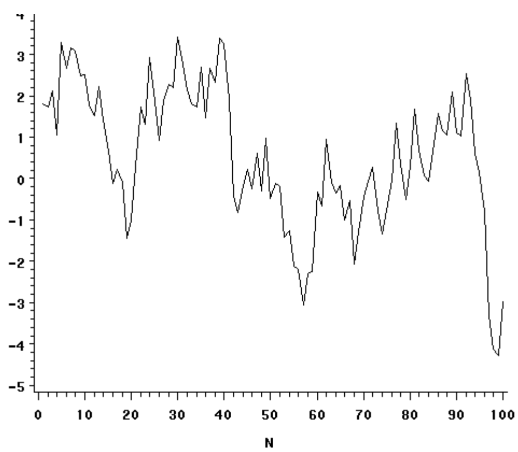
An example realization of a Gaussian white noise process.

Time series Models

- Random walk Process
  - When $\{z_t\}$ is a discrete purely random process with mean μ and variance σ^2 a process $\{x_t\}$ is said to be a random walk if,

    $x_t = x_{t-1} + z_t$.

That is, the change of y is absolutely random.



Random walk

Generalized linear processes

- When $\{x_t\}$ is a purely random process with mean zero, the process $\{x_t\}$ is said to be a general linear process if,

  $x_t = \beta_0 z_t + \beta_1 z_{t-1} + ...$

  Where $\beta_i s$ are constants.

- **Moving Average Process**

  Let $\{z_t\}$ be a purely random process with mean zero variance σ^2 then process $\{x_t\}$ is said to be a moving average process of order q (MA(q))

If $x_t = \beta_0 z_t + \beta_1 z_{t-1} + ... + \beta_q z_{t-q}$. Where $\beta_i s$ are constants.

Order of a MA process can be assessed by looking for the lag beyond which the sample ACF is not significantly zero.

- **Invertibility of MA(q) process**

$$X_t = z_t + \beta_1 z_{t-1} + \beta_2 z_{t-2} + ... + \beta_q z_{t-q}$$

Using Lag (Backward) operator it can be shown that

$$X_t = (1 + \beta_1 L + \beta_2 L^2 + ... + \beta_q L^q) z_t$$

$$= \theta(L) z_t$$

$\theta(y) = 0$ is called the characteristic equation.

MA(q) process is invertible if the roots of the characteristic equation lie outside the unit cirle . (Box & Jenkins).

- **<u>MA(1) Process</u>**

$$X_t = z_t + \beta_1 z_{t-1}$$

$X_t = (1 + \beta_1 L) z_t \Rightarrow$ Characteristic equation is $1 + \beta_1 y = 0$

MA(1) process is invertible if $y = -\frac{1}{\beta_1}$ lie outside the unit cirle .

i.e. if $\left|\frac{1}{\beta_1}\right| > 1$

Then $|\beta_1| < 1$.

Also It can be shown that $z_t = \sum_{i=0}^{\infty} (-\beta_1)^i X_{t-i}$

This series converges if $|\beta_1| < 1$.

Therefore convergence of the series is also known as the invertibility

Autoregressive (AR) processes

- If $\{Z_t\}$ is a purely random process with mean 0 and variance σ^2 then the process is said to be an autoregressive process of order p if

  $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ...+ \alpha_p X_{t-p} + Z_p$

  Where $\alpha_1, \alpha_2, ..., \alpha_p$ are constants.

- **AR(1) process (Markov Process)**
  - AR(1) is an infinite order MA process
  - $E(X_t) = 0$
  - $V(X_t) = \sigma^2 \sum_{i=0}^{\infty} \alpha_1^{2i}$

    -variance is finite provided that $|\alpha_1| < 1$.
  - The Autocorrelation function $\rho(k) = \alpha_1^{|k|}, \quad k = 0, \pm 1, \pm 2.....$

- **AR(p) Process**
  - AR(p) model
    - $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ...+ \alpha_p X_{t-p} + Z_t$

      $Z_t = X_t - \alpha_1 X_{t-1} - \alpha_2 X_{t-2} - ...... - \alpha_p X_{t-p}$
  - Using Lag operation

$$Z_t = (1-\alpha_1 L - \alpha_2 L^2 - .... - \alpha_p L^p) X_t$$

- AR(p) process is stationary if all the roots of the characteristic equation lie outside the unit circle.

$E(X_t) = 0$

- $\gamma(k) = Cov(X_t, X_{t-k})$
  $= \alpha_1 \gamma(k-1) + \alpha_2 \gamma(k-2) + ... + \alpha_p \gamma(k-p) + Z_t$

- $\rho(k) = \alpha_1 \rho(k-1) + \alpha_2 \rho(K-2) + ... + \alpha_p \rho(k-p)$

- As k varies a set of equations are formed. These equations are called Yule-Walker equations.

- The Yule-Walker equations has a general solution as

$$\rho(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|} + ... + A_p \pi_p^{|k|}$$

where π_i s arethe roots of the auxiliary equation

$y^p - \alpha_1 y^{p-1} - .... - \alpha_p = 0$ and A_i s are selected in a way that

$$\rho(0) = 1 \quad and \quad \sum_{i=1}^{p} A_i = 1 .$$

- $\rho(k)$ tends to zero as k increases provided that $|\pi_i| < 1$. The necessary and sufficient condition for an autoregressive process to be stationary is $|\pi_i| < 1$ for all i.

- Partial Autocorrelation Function

- When fitting an AR(p) model the last coefficient α_p measures the acces correlation at lag p which is not accounted by an AR(p-1) model. It is called p^th^ partial autocorrelation coefficient.

- The plot of partial autocorrelation coefficients plotted against lag p can be used to determine the order of an AR process.

- The values of α_p which lie outside the range $\pm 2/\sqrt{N}$ are significantly different from zero.
- The partial ACF of an AR(p) process 'cuts off' at lag p.

The following table summarizes how we use the sample autocorrelation function for model identification.

SHAPE	**INDICATED MODEL**
Exponential, decaying to zero	Autoregressive model. Use the partial autocorrelation plot to identify the order of the autoregressive model.
Alternating positive and negative, decaying to zero	Autoregressive model. Use the partial autocorrelation plot to help identify the order.
One or more spikes, rest are essentially zero	Moving average model, order identified by where plot becomes zero.
Decay, starting after a few lags	Mixed autoregressive and moving average model.
All zero or close to zero	Data is essentially random.
High values at fixed intervals	Include seasonal autoregressive term.
No decay to zero	Series is not stationary.

Box-Jenkins Model Identification

<u>Stationarity and Seasonality</u>

The first step in developing a Box-Jenkins model is to determine if the series is stationary and if there is any significant seasonality that needs to be modeled. *Stationarity* can be detected from an Autocorrelation Plot. Specifically, non-stationarity is often indicated by an autocorrelation plot with very slow decay.Seasonality (or periodicity) can also be assessed from an Autocorrelation plot , a time series plot.

Box and Jenkins recommend the differencing approach to achieve stationarity.

Seasonal differencing

This may help in the model idenfitication of the non-seasonal component of the model. In some cases, the seasonal differencing may remove most or all of the seasonality effect.

Identify p and q

Once stationarity and seasonality have been considered, the next step is to identify the order (i.e., the ***p*** and ***q***) of the autoregressive and moving average terms.

Autocorrelation and Partial Autocorrelation Plots

The primary tools for doing this are the Autocorrelation plot and the Partial Autocorrelation Plot.

The sample autocorrelation plot and the sample partial autocorrelation plot are compared to the theoretical behavior of these plots when the order is known.

Order of Autoregressive Process (p)

Specifically, for an AR(1) process, the sample autocorrelation function should have an exponentially decreasing appearance. However, higher-order AR processes are often a mixture of exponentially decreasing and damped sinusoidal components.

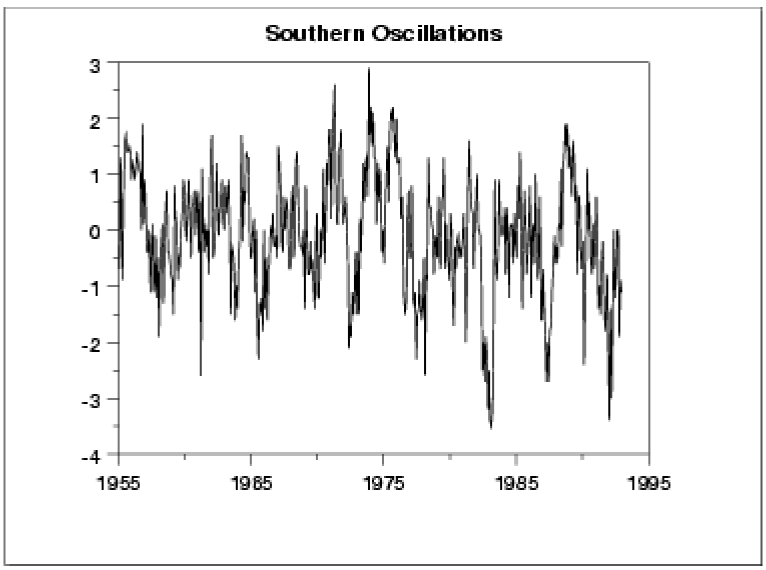
For higher-order autoregressive processes, the sample autocorrelation needs to be supplemented with a partial autocorrelation plot. The partial autocorrelation of an AR(***p***) process becomes zero at lag ***p*+1** and greater, so we examine the sample partial autocorrelation function to see if there is evidence of a departure from zero. This is usually determined by placing a 95% confidence interval on the sample partial autocorrelation plot.

Order of Moving Average Process (q)

The autocorrelation function of a MA(q) process becomes zero at lag $q+1$ and greater, so We examine the sample autocorrelation function to see where it essentially becomes zero. We do this by placing the 95% confidence interval for the sample autocorrelation function on the sample autocorrelation plot.

The sample partial autocorrelation function is generally not helpful for identifying the order of the moving average process.

Example: For the southern Oscillations data set.

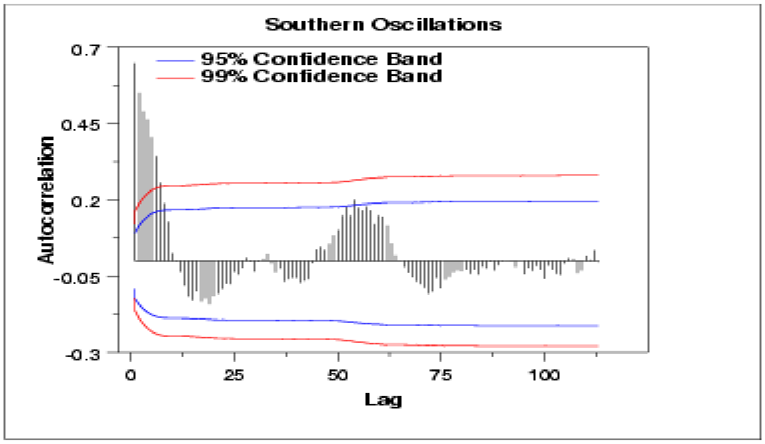


run sequence plot

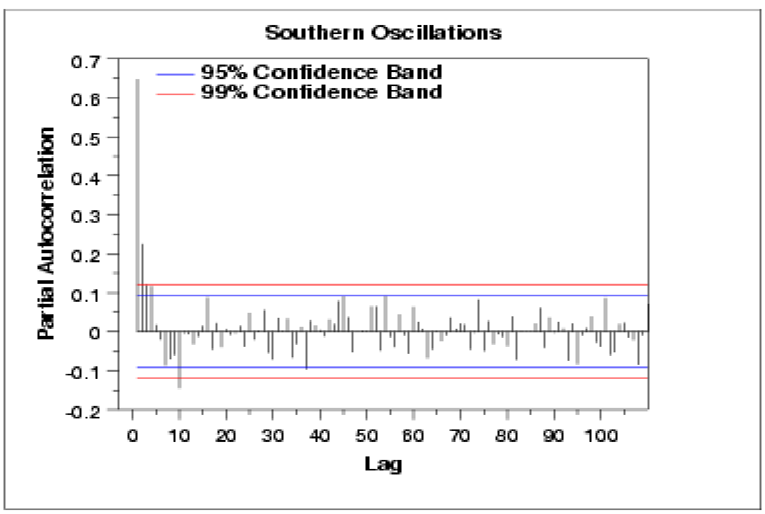
The run sequence plot indicates stationarity.

Since the above plots show that this series does not exhibit any significant non-stationarity or seasonality, we generate the autocorrelation and partial autocorrelation plots of the raw data.

Autocorrelation plot



The autocorrelation plot shows a mixture of exponentially decaying and damped sinusoidal components. This indicates that an autoregressive model, with order greater than one, may be appropriate for these data. The partial autocorrelation plot should be examined to determine the order.



partial autocorrelation plot

The partial autocorrelation plot suggests that an AR(2) model might be appropriate.

In summary, our intial attempt would be to fit an AR(2) model with no seasonal terms and no differencing or trend removal. Model validation should be performed before accepting this as a final model.

Mixed Models

- A useful class of time series formed from a combination of MA and AR processes.
- Examples- ARMA and ARIMA models
- An autoregressive moving average process of order (p,q) denoted by ARMA(p,q) is given by

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + ... + \alpha_p X_{t-p} + Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2} + ... + \beta_q Z_{t-q}$$

Where α_i and β_i for i= 1,2… are constants and $\{Z_t\}$ is a purely random process.

- ARMA models

- ARMA(p,q) model can be represented in the form $\phi(L)X_t = \theta(L)Z_t$

Where $\phi(L) = 1 - \alpha_1 L - \alpha_2 L^2 - .... - \alpha_p L^p$

$$\theta(L) = 1 + \beta_1 L + \beta_2 L^2 + ... + \beta_q L^q$$

- The process is stationary if the roots of the equation $\phi(y) = 0$ exceed 1 in modulus.
- It is invertible if the roots of $\theta(y) = 0$ exceed 1 in modulus.
- Correlogram and the plot of partial autocorrelation function against lagcan be used to determine the order of an ARMA(p,q) process

  - Plot of PACF against lag cuts off at lag p

  - Correlogram cuts off at lag q

Diagnostic checking

When a model has been fitted to a time series , it is advisable to check the model diagnostics.

This is usually done by looking at the residuals of the fitted model.

Residual = observation – fitted value.

If we have a good model then we expect the residuals to be random and close to zero and are normally distributed.

(1) Randomness

- Residual Correlogram

This is used to examine autocorrelation effect more closely.

For a good model we expect residual to behave as white noise. The residual which lie outside the range $\pm 1.96/\sqrt{n}$ (n is the sample size) are significantly different from zero at 5% level means the evidence of wrong modelhas been fitted.

- Ljung-Box Chi squared statistics

H_0 : Residuals are white noise.(model is adequate)

H_1_: Not so

Test statistics

$Q = n(n+2)\sum_{r=1}^{h} \frac{{\rho_h}^2}{n-r}$ where ρ_h is the Autocorrelation function at lag h.

Under H_0 $Q \approx \chi^2_{h-(p+q)}$. If $Q < \chi^2_{h-(p+q),\alpha\%}$ then we accept H_0 and conclude that the residuals are white noise.

(2) Normality

Bell shape in histogram or straight line pattern in normal probability plot suggests the normality of residuals.

(3) Outliers and regular pattern

Absence of any pattern in the Plot of residual vs predicted values indicates a good fit.

The time plot of residuals will also reveal any regular pattern and outliers.