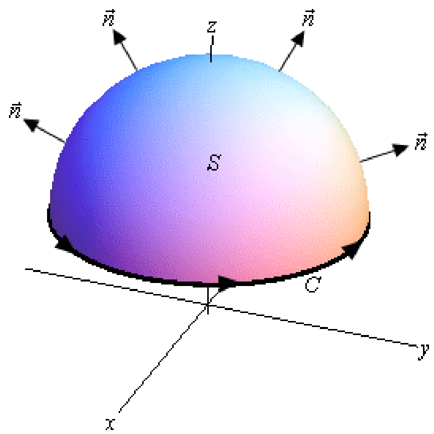
Stokes' Theorem

In this section we are going to take a look at a theorem that is a higher dimensional version of Green's Theorem. In this section we are going to relate a line integral to a surface integral. However, before we give the theorem we first need to define the curve that we're going to use in the line integral.
Let's start off with the following surface with the indicated orientation. Around the edge of this surface we have a curve C. This curve is called the boundary curve.
The orientation of the surface S will induce the positive orientation of C.



Stokes' Theorem
Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\underline{F}$ be a vector field then,

$$\int_C \underline{F}.d\underline{r} = \iint_S curl\,\underline{F}.d\underline{s}$$

In this theorem note that the surface *S* can actually be any surface so long as its boundary curve is given by *C*. This is something that can be used to our advantage to simplify the surface integral on occasion.
Let's take a look at a couple of examples.

Example 1: Verify the Stokes' theorem for $\underline{A} = 3y\underline{i} - xz\,\underline{j} + yz^2\,\underline{k}$ where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary.

Example 2: Use Stokes' Theorem to evaluate $\int_C \underline{F}.d\underline{r}$ where $\underline{F} = z^2\underline{i} + y^2\,\underline{j} + x\underline{k}$ and *C* is the triangle with vertices $(1, 0, 0)$, $(0,1,0)$ and $(0, 0,1)$ with counter-clockwise rotation.

In both of these examples we were able to take an integral that would have been somewhat unpleasant to deal with and by the use of Stokes' Theorem we were able to convert it into an integral that wasn't too bad.

Divergence Theorem

In this section we are going to relate surface integrals to triple integrals. We will do this with the Divergence Theorem.

Divergence Theorem
Let *E* be a simple solid region and *S* is the boundary surface of *E* with positive orientation. Let $\underline{F}$ be a vector field whose components have continuous first order partial derivatives. Then, $\iint_S \underline{F}.d\underline{s} = \iiint_E div\underline{F}\,dV$

Let's see an example of how to use this theorem.

Example 1: Verify the divergence theorem for $\underline{A} = (2x - z)\underline{i} + x^2 y\underline{j} - xz^2\underline{k}$ taken over the region bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Example 2: Evaluate $\iint_S xz^2 dydz + (x^2 y - z^3)dzdx + (2xy + y^2 z)dxdy$ where S is the entire surface of the hemispherical region bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ by the divergence theorem.