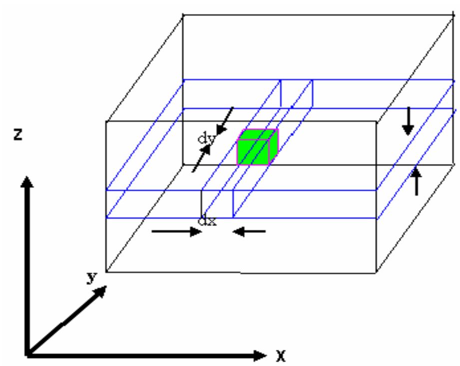
Triple Integrals

In principle triple integral are the same as double integrals, with an extra unknown. Triple integrals are also known as **Volume Integrals.** Consider a solid body occupying a volume V in 3D. Let its density by $\rho(x, y, z)$. The mass of a small element $\Delta x \Delta y \Delta z$ is $\rho(x, y, z)\Delta x \Delta y \Delta z$



Hence the total mass of the solid is $M = \iiint_V \rho(x, y, z)dxdydz$ best illustrated by example

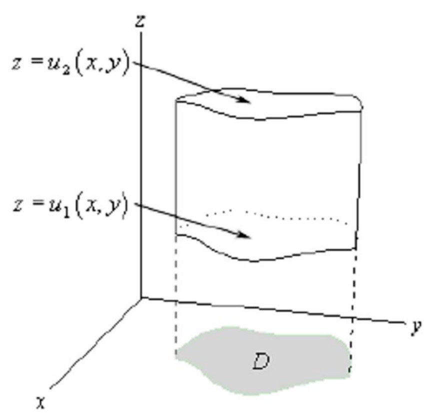
Example 1 Evaluate the following integral.

$$\iiint_B 8xyz\, dV, \quad B = [2,3]\times[1,2]\times[0,1]$$

Before moving on to more general regions letôs get a nice geometric interpretation about the triple integral out of the way so we can use it in some of the examples to follow.

Fact : The volume of the three-dimensional region *E* is given by the integral, $V = \iiint_E dV$

Letôs now move on the more general three-dimensional regions. We have three different possibilities for a general region. Here is a sketch of the first possibility.



In this case we define the region *E* as follows, $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$

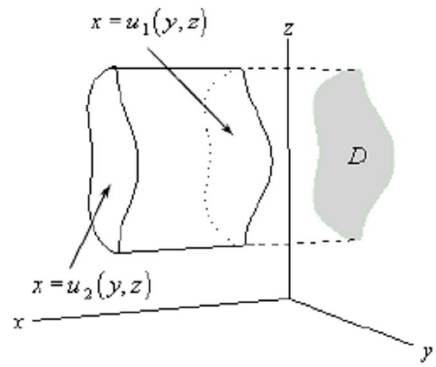
where $(x,y)\in D$ is the notation that means that the point (x, y) lies in the region *D* from the *xy*-plane. In this case we will evaluate the triple integral as follows,

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

where the double integral can be evaluated in any of the methods that we saw in the previous couple of sections. In other words, we can integrate first with respect to *x*, we can integrate first with respect to *y*, or we can use polar coordinates as needed.

Example 2 Evaluate $\iiint\limits_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.

Letôs now move onto the second possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



For this possibility we define the region *E* as follows,

$$E=\{(x,y,z)\,|\,(y,z)\in D,\ u_1(y,z)\le x\le u_2(y,z)\}$$

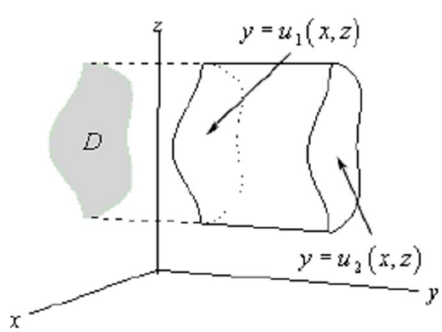
So, the region *D* will be a region in the *yz*-plane. Here is how we will evaluate these integrals.

$$\iiint\limits_E f(x,y,z)\,dV=\iint\limits_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right]dA$$

As with the first possibility we will have two options for doing the double integral in the *yz*-plane as well as the option of using polar coordinates if needed.

Example 3 Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the *yz*-plane that is bounded by $z=\frac{3}{2}\sqrt{y}$ and $z=\frac{3}{4}y$.

We now need to look at the third (and final) possible three-dimensional region we may run into for triple integrals. Here is a sketch of this region.



In this final case *E* is defined as,

$$E=\{(x,y,z)\,|\,(x,z)\in D,\ u_1(x,z)\le y\le u_2(x,z)\}$$

and here the region *D* will be a region in the *xz*-plane. Here is how we will evaluate these integrals.

$$\iiint\limits_E f(x,y,z)\,dV=\iint\limits_D\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right]dA$$

Where we will can use either of the two possible orders for integrating *D* in the *xz*-plane or we can use polar coordinates if needed.

Example 4 Evaluate $\iiint_E \sqrt{3x^2+3z^2}\, dV$ where E is the solid bounded by $y = 2x^2 + 2z^2$ and the plane $y = 8$.

Triple Integrals in Cylindrical Coordinates

In this section we want do take a look at triple integrals done completely in Cylindrical Coordinates. The cylindrical coordinates are really nothing more than an extension of polar coordinates into three dimensions. The following are the conversion formulas for cylindrical coordinates. $x = r\cos\theta \quad y = r\sin\theta$ z=z

In order to do the integral in cylindrical coordinates we will need to know what *dV* will become in terms of cylindrical coordinates. We will be able to show in the Change of Variables section of this chapter that, $dv = rdzdrd\theta$

The region, *E*, over which we are integrating becomes,

$$E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$$
$$= \{(r,\theta,z) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta),\ u_1(r\cos\theta, r\sin\theta) \le z \le u_2(r\cos\theta, r\sin\theta)\}$$

Note that weŏve only given this for *E*ŏs in which *D* is in the *xy*-plane. We can modify this accordingly if *D* is in the *yz*-plane or the *xz*-plane as needed.
In terms of cylindrical coordinates a triple integral is,

$$\iiint_E f(x,y,z)dv = \int_\alpha^\beta \int_{h1(\theta)}^{h2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} rf(r\cos\theta, r\sin\theta, z)dzdrd\theta$$

Donŏt forget to add in the *r* and make sure that all the *x*ŏs and *y*ŏs also get converted over into Cylindrical coordinates. Letŏs see an example.

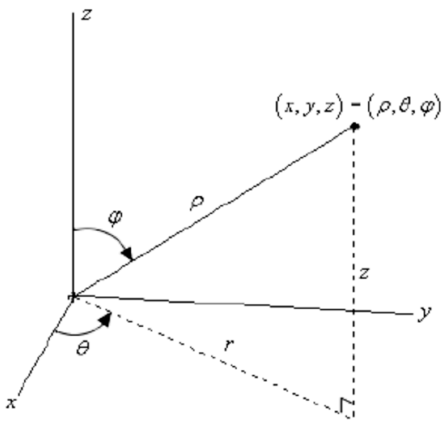
Example 1 Evaluate $\iiint_E y\, dV$ where E is the region that lies below the plane $z = x + 2$ ab the *xy*-plane and between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Just as we did with double integral involving polar coordinates we can start with an iterated integral in terms of *x*, *y*, and *z* and convert it to cylindrical coordinates.

Example 2 Convert $\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$ into an integral in cylindrical coordinates.

Triple Integrals in Spherical Coordinates

In the previous section we looked at doing integrals in terms of cylindrical coordinates and we now need to take a quick look at doing integrals in terms of spherical coordinates.
First, we need to know how spherical coordinates are defined. The following sketch shows the relationship between the Cartesian and spherical coordinate systems.



Here are the conversion formulas for spherical coordinates.

$$x = \rho \sin\varphi \cos\theta \qquad y = \rho \sin\varphi \sin\theta \qquad z = \rho\cos\varphi$$

$$x^2 + y^2 + z^2 = \rho^2$$

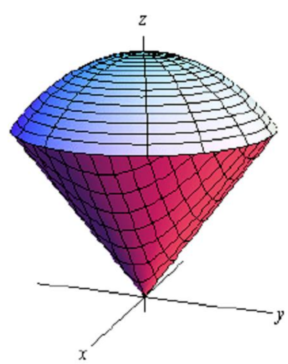
We also have the following restrictions on the coordinates.

$$\rho \geq 0 \qquad 0 \leq \varphi \leq \pi$$

For our integrals we are going to restrict *E* down to a spherical wedge. This will mean that we are going to take ranges for the variables as follows,

$$a \leq \rho \leq b$$
$$\alpha \leq \theta \leq \beta$$
$$\delta \leq \varphi \leq \gamma$$

Here is a quick sketch of a spherical wedge in which the lower limit for both ρ and φ are zero for reference purposes. Most of the wedges weôll be working with will fit into this pattern.



From this sketch we can see that *E* is really nothing more than the intersection of a sphere and a cone. In the next section we will show that

$$dv = \rho^2 \sin\varphi d\rho d\theta d\varphi$$

Therefore the integral will become,

$$\iiint_E f(x,y,z)dv = \int_a^b \int_\alpha^\chi \int_\delta^\gamma \rho^2 \sin\varphi f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) d\rho d\theta d\varphi$$

This looks bad, but given that the limits are all constants the integrals here tend to not be too bad.

Example 1 Evaluate $\iiint_E 16z\, dV$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$.

Example 2 Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 \, dz\, dx\, dy$ into spherical coordinates.